Queen Mary
University of London

B. Sc. Examination by course unit 2011

MTH6132 Relativity

Duration: 2 hours

Date and time: 3rd May 2011, 10:00 am

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): Dr. Juan A. Valiente Kroon

You are reminded of the following information, which you may use without proof.

- Lower case Latin indices run from 0 to 3.
- The metric tensor of the Minkowski spacetime is η_{ab} such that

$$\mathrm{d}s^2 = \eta_{ab}\mathrm{d}x^a\mathrm{d}x^b = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

- The Lorentz transformations between two frames F and F' in standard configuration are given by

$$x' = \gamma(x - vt), \quad t' = \gamma\left(t - \frac{vx}{c^2}\right), \quad y' = y, \quad z' = z$$

where

$$\gamma = \frac{1}{\sqrt{1 - (v^2/c^2)}}$$

and F' is moving with speed v relative to F.

- The covariant derivative of a covariant vector is given by

$$\nabla_a V_b = \partial_a V_b - \Gamma^f{}_{ba} V_f.$$

- The metric tensor satisfies:

$$g_{ab}g^{bc} = \delta_a{}^c.$$

- Christoffel symbols (connection):

$$\Gamma^m{}_{ij} = \frac{1}{2}g^{mk}(\partial_i g_{kj} + \partial_j g_{ik} - \partial_k g_{ij}).$$

- The Riemann curvature tensor:

$$R^a{}_{bcd} = \partial_c\Gamma^a{}_{bd} - \partial_d\Gamma^a{}_{bc} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}.$$

- Euler–Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{x}^c}\right) - \frac{\partial L}{\partial x^c} = 0$$

- Geodesic equations:

$$\ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0.$$

SECTION A: *You should attempt all questions. Marks awarded are shown next to the questions.*

Question 1

(i) On page 2 you are given the Lorentz transformation that allows to write the coordinates (t', x', y', z') in terms of the coordinates (t, x, y, z). Write the inverse transformation (that is, the transformation that allows to write the coordinates (t, x, y, z) coordinates in terms of the coordinates (t', x', y', z')). Justify your answer.

(ii) Show that

$$-c^2t^2 + x^2 + y^2 + z^2$$

is an invariant upon the Lorentz transformation given on page 2.

(iii) Draw a 2-dimensional —i.e. (t, x)— spacetime diagram describing the following situation: a light ray is shot from the origin at time $t = t_1$; it hits a mirror located at $x = x_0$ and it is reflected back to the origin; it reaches the origin at a time $t = t_2$. To draw the diagram use units for which $c = 1$. At what time does the light hit the mirror?

[10 Marks]

Question 2

In what follows, let $\bar{A}$ and $\bar{B}$ denote two arbitrary 4-vectors.

(i) Define what is meant by the norm $|\bar{A}|^2$ and the scalar product $\bar{A} \cdot \bar{B}$.

(ii) What are the conditions for $\bar{A}$ to be timelike, spacelike and null?

(iii) Using the fact that $|\bar{A}|^2$, $|\bar{B}|^2$ and $|\bar{A} + \bar{B}|^2$ are invariants, show that the scalar product $\bar{A} \cdot \bar{B}$ is also an invariant.

[10 Marks]

Question 3

(i) Define the Kronecker delta $\delta_a{}^b$. Show that in 4 dimensions one has that

$$\delta_a{}^a = 4.$$

(ii) Give the transformation rule of a type $(2, 2)$ tensor $S^{ab}{}_{cd}$. Show that $S^{ab}{}_{ca}$ is a tensor of type $(1, 1)$ and that $S^{ab}{}_{ba}$ is a scalar —i.e. a tensor of type $(0, 0)$.

(iii) Show that if W_{ab} is a tensor of type $(0, 2)$ satisfying

$$W_{ab} = W_{(ab)}$$

in one system of coordinates, then $W_{ab} = W_{(ab)}$ for all systems of coordinates.

[10 Marks]

Question 4

The metric for a particular two-dimensional spacetime is given by

$$\mathrm{d}s^2 = -y^3\mathrm{d}x^2 + x^4\mathrm{d}y^2$$

(i) Calculate by the method you prefer all the components of the connection $\Gamma^a{}_{bc}$ for this metric.

(ii) Calculate the $R^x{}_{yxy}$ component of the Riemann tensor.

[10 Marks]

Question 5

The line element and the metric tensor are related to each other via

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b.$$

(i) By recalling that $\mathrm{d}s^2$ is an invariant and that $\mathrm{d}x^a$, $\mathrm{d}x^b$ are arbitrary, show that g_{ab} is a type $(0,2)$ tensor.

(ii) The line element of the Minkowski metric in Cartesian coordinates is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

Show that the Minkowski metric η_{ab} given by the above line element is a solution to the vacuum Einstein field equations

$$R_{ab} = 0.$$

(iii) Why is the metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2,$$

also a solution to the vacuum Einstein field equations?

[10 Marks]

SECTION B: *Each question carries 25 marks. You may attempt all questions. Except for the award of a bare pass, only marks for the best TWO questions will be counted.*

Question 6

In this question consider units for which $c = 1$.

(i) Define the 4-velocity $\bar{u}$ of a timelike particle in Special Relativity. Show that for such particle

$$|\bar{u}|^2 = -1.$$

Define the 4-momentum $\bar{p}$. Prove that

$$|\bar{p}|^2 = -m_0^2,$$

where m_0 is the rest mass of the particle.

(ii) Show that the 4-momentum can be expressed in terms of the 3-velocity of the particle, $\underline{v}$, and its rest mass as

$$\bar{p} = m_0\gamma(v)(1, \underline{v}),$$

where

$$\gamma(v) = \frac{1}{\sqrt{1-v^2}}, \quad v = |\underline{v}|.$$

(iii) An atom of rest mass m_0 at rest in a laboratory absorbs a photon of frequency ν. Use the conservation law of 4-momentum to find the velocity and rest mass of the resulting particle.

Question 7

In this question ∇_a denotes the covariant derivative given by the Levi-Civita connection (Christoffel symbols) as given on page 2.

(i) On page 2 you are given the formula for the covariant derivative of a covariant tensors of type $(0, 1)$. What is the corresponding formula for the covariant derivative of a rank 2 covariant tensor —$(0, 2)$ tensor?

(ii) Recalling that for a scalar, ϕ, one has that $\nabla_a\phi = \partial_a\phi$, show that

$$\nabla_a\nabla_b\phi = \nabla_b\nabla_a\phi.$$

(iii) Show, by a direct computation, that

$$\nabla_a g_{bc} = 0.$$

Question 8

Consider the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \,\mathrm{d}\varphi^2\right).$$

(i) Derive the geodesic equations satisfied by a photon for this metric using the Euler-Lagrange equations.

(ii) Derive the equation expressing that the 4-velocity of a photon moving on the Schwarzschild spacetime has zero norm.

(iii) Use the geodesic equations to find an expresion for $\mathrm{d}r/\mathrm{d}t$. Show that the path for a light ray moving radially inwards in the Schwarzschild metric is given by the condition

$$t + r + 2GM \ln(r - 2GM) = \text{constant}$$

End of Paper