Relativistic Astrophysics. 2007. Course Work 1. Solutions

Q1.

a) Following the original Laplace calculation within the framework of Newtonian gravity, show that the escape velocity from the surface of a gravitating body of mass M is equal to the speed of light, if the radius of the body is equal to $2GM/c^2$ (gravitational radius).

The escape velocity from the surface of this body of mass M and of radius R is

$$v_{esc} = \sqrt{\frac{2GM}{R}}.$$

The escape velocity is equal to the speed of light

$$v_{esc} = c$$

if

$$R = r_g \equiv \frac{2GM}{c^2}.$$

b) A star forms a black hole of mass M. Show that to an order of magnitude its density at the moment immediately before the formation of the black hole is

$$2 \times 10^{16}\ g \cdot cm^{-3} \left(\frac{M}{M_\odot}\right)^{-2}.$$

For what mass is this equal to the density of air ($\approx 10^{-5} g/cm^3$)?

To an order of magnitude

$$\rho = \frac{M}{V} = \frac{M}{\frac{4\pi}{3} r_g^3} = \frac{3M}{4\pi \left(\frac{2GM}{c^2}\right)^3} = \frac{3M_\odot}{4\pi \left(\frac{2GM_\odot}{c^2}\right)^3} \left(\frac{M}{M_\odot}\right)^{-2} =$$

$$= \frac{3}{4\pi} \frac{M_\odot}{(3\text{ km})^3} \left(\frac{M}{M_\odot}\right)^{-2} = \frac{3 \cdot 2 \times 10^{30}\text{ kg}}{4 \cdot 3.14 \cdot 3^3 \times 10^9\text{ m}^3} \left(\frac{M}{M_\odot}\right)^{-2} \approx 2 \times 10^{19} \text{kg} \cdot \text{m}^{-3} \left(\frac{M}{M_\odot}\right)^{-2}.$$

If this density is equal to the density of air, then

$$\rho = 10^{-5} g/cm^3 = \frac{10^{-5} \cdot 10^{-3}\text{kg}}{10^{-6}\text{m}^3} = 10^{-2}\text{kg} \cdot \text{m}^{-3},$$

hence

$$10^{-2}\text{kg} \cdot \text{m}^{-3} = 2 \times 10^{19}\text{kg} \cdot \text{m}^{-3} \left(\frac{M}{M_\odot}\right)^{-2}$$

and

$$\frac{M}{M_\odot} \approx \left(2 \cdot 10^{21}\right)^{1/2} \approx 4 \cdot 10^{10}.$$

The final answer is

$$M = 4 \cdot 10^{10} M_\odot.$$

Q2.

a) Consider a photon with energy $E = h\nu$ climbing out of a gravitational field and use energy conservation law to show that in traveling through a potential difference $\delta U \ll c^2$, the photon should experience a fractional frequency shift

$$z = -\frac{\delta U}{c^2}.$$

An effective gravitational mass of a photon of frequency of ν is

$$m_{photo} = \frac{E_{photon}}{c^2} = \frac{h\nu}{c^2}.$$

From conservation of energy we have

$$h\nu_{ob} - \frac{Gm}{R_{ob}}\frac{h\nu_{ob}}{c^2} = h\nu_{em} - \frac{Gm}{R_{em}}\frac{h\nu_{em}}{c^2},$$

where "ob" corresponds to observation and "em" to emission of the photon. Thus

$$\frac{\nu_{ob}}{\nu_{em}} = \frac{1 - \frac{Gm}{R_{em}c^2}}{1 - \frac{Gm}{R_{ob}c^2}} \approx 1 - \frac{Gm}{R_{em}c^2} + \frac{Gm}{R_{ob}c^2}.$$

Taking into account that in Newtonian limit

$$\delta U = \frac{Gm}{R_{ob}} - \frac{Gm}{R_{em}} < 0,$$

we have

$$z = \frac{|\nu_{obs} - \nu_{em}|}{\nu_{em}} = 1 - \left[1 - \frac{Gm}{R_{em}c^2} + \frac{Gm}{R_{ob}c^2}\right] = -\frac{\delta U}{c^2}.$$

b) From observations of some unknown object it was found that a fractional frequency shift of spectral lines was equal to

$$z = \frac{|\delta\nu|}{\nu} = (3.2 \pm 1.2) \cdot 10^{-4}.$$

Assuming that this redshift is explained by the redshift in the gravitational field of a solar mass object, calculate the predictable redshifts caused by a star of solar type, a white dwarf and a neutron star, whose typical radii may be taken to be 700000 km, 6000 km and 10 km, respectively. Hence determine which type of object was observed.

For a star of solar type

$$z \approx \frac{GM}{rc^2} = \frac{r_{g\odot}}{2r}\left(\frac{M}{M\odot}\right) \approx \frac{3}{2 \cdot 7 \cdot 10^5} \approx 2 \cdot 10^{-6};$$

For a white dwarf

$$z \approx \frac{GM}{rc^2} = \frac{r_{g\odot}}{2r}\left(\frac{M}{M\odot}\right) \approx \frac{3}{2 \cdot 6 \cdot 10^3} \approx 2.5 \cdot 10^{-4};$$

For a neutron star

$$z \approx \frac{GM}{rc^2} = \frac{r_{g\odot}}{2r}\left(\frac{M}{M\odot}\right) \approx \frac{3}{2 \cdot 10} \approx 0.15.$$

The final answer is: a white dwarf was observed.

Q3.

a) Using simple Newtonian estimates, show that the radius of tidal disruption, R_{TD}, for a star of mass m and radius r in the gravitational field of the black hole of mass M is

$$R_{TD} \approx r \left(\frac{M}{m}\right)^{1/3}.$$

Self-gravity force is

$$F_{sg} \approx Gm\delta m/r^2.$$

The tidal force is

$$F_{TD} \approx GM\delta mr/R^3.$$

The tidal radius is determined from

$$F_{sg} \approx F_{TD},$$

hence

$$\frac{Gm\delta m}{r^2} \approx \frac{GM\delta mr}{R_{TD}^3}$$

and, finally,

$$R_{TD} \approx r(M/m)^{1/3}.$$

b) Find the critical value of the black hole mass, M_{crit}, for which R_{TD} equals the gravitational radius of the black hole.

The tidal radius for the solar type star is equal to the gravitational radius if

$$R_{TD} = r_\odot \left(\frac{M}{M_\odot}\right)^{1/3} = r_g = \frac{2GM}{c^2} = 3\ km \frac{M}{M_\odot},$$

hence

$$r_\odot \left(\frac{M_{cr}}{M_\odot}\right)^{1/3} = r_{g\odot} \left(\frac{M_{cr}}{M_\odot}\right),$$

hence

$$\left(\frac{M_{cr}}{M_\odot}\right)^{2/3} \approx \left(\frac{r_\odot}{r_{g\odot}}\right)$$

and, finally,

$$M_{cr} \approx M_\odot \left(\frac{r_\odot}{r_{g\odot}}\right)^{3/2} \approx \left(\frac{7 \cdot 10^5}{3}\right)^{3/2} \approx 10^8 M_\odot.$$

Relativistic Astrophysics. 2007. Course Work 2. Solutions

Q1.

Formulate the equivalence principle and explain what is the difference in interpretation of this principle in Newtonian theory and in General relativity.

This principle states that an uniform gravitational field is equivalent to a uniform acceleration of reference frame.

In Newton theory the motion of a test particle is determined by the following equation of motion

$$m_{in}\vec{a} = -m_{gr}\nabla\phi,$$

where $\vec{a}$ is the acceleration of the test particle, ϕ is newtonian potential of gravitational field, m_{in} is the inertial mass of the test particle and m_{gr} is its gravitational mass. The fact that all test particles move with the same acceleration for given ϕ is explained within frame of newtonian theory just by the following "coincidence":

$$\frac{m_{in}}{m_g} = 1,$$

i.e. inertial mass m_{in} is equal to gravitational mass m_{gr}.

The General Relativity gives very simple and natural explanation of the Principle of Equivalence: In curved space-time all bodies move along geodesics, that is why their world lines are the same in given gravitational field. The situation is the same as in flat space-time when free particles move along straight lines which are geodesics in flat space-time. What is geodesic we will also discuss the next lecture.

Q2.

A rocket moves very far from any gravitating bodies with acceleration $5g$. Using the equivalence principle, show, that in first order with respect to h/R the redshift of a photon emitted at the bottom of the rocket and detected at its top is the same as if the rocket were at rest on the surface of a planet with mass M and radius R related by the following relationship: $MR_{\oplus}^2 = 5M_{\oplus}R^2$. Calculate the redshift if the height of the rocket is $169m$. (You can assume that the diameter of the Earth is 13 000 km and its gravitational radius is 1 cm).

The non-inertial frame of reference moving with acceleration $5g$ is equivalent to gravitational field of mass M and radius R if

$$\frac{GM}{R^2} = 5g = \frac{5GM_{\oplus}}{R_{\oplus}^2},$$

hence

$$MR_{\oplus}^2 = 5M_{\oplus}R^2.$$

From solution to question Q2 of CW1 we know that the redshift is

$$z = \frac{\delta U}{c^2},$$

where

$$U = -\frac{GM}{R}$$

and

$$\delta U = \delta R \frac{dU}{dR} = h\frac{GM}{R^2};$$

hence

$$z = \frac{GMh}{R^2c^2} = \frac{5GM_{\oplus}h}{R_{\oplus}^2} = \frac{5r_{\oplus g}h}{R_{\oplus}^2} \approx \frac{5 \cdot 10^{-2}m \cdot 169m}{(1310^6m)^2} = 5 \cdot 10^{-14}.$$

Q3.

Formulate the covariance principle and explain the relationship between this principle and the principle of equivalence.

This principle says: The shape of all physical equations should be the same in an arbitrary frame of reference, including the most general case of non-inertial frames. If in contrast to the covariance principle the shape of physical equations were different in local inertial frames in presence of gravitational field and in non-inertial frames in absence of gravitational field then these equations would give different solutions, i.e. different predictions for (a) standing on the Earth, feeling the effects of gravity as a downward pull and (b) standing in a very smooth elevator that is accelerating upwards with the acceleration g , hence these equations would contradict to the basic postulate of the General Relativity, the principle of equivalence, which states that a uniform gravitational field (like that near the Earth) is equivalent to a uniform acceleration. Hence, the covariance principle is the mathematical formulation of the principle of equivalence.

Relativistic Astrophysics. 2007. Course Work 3. Solutions

Q1.

a) Explain what is the reciprocal tensor.

Two tensors A_{ik} and B^{ik} are called reciprocal to each other if

$$A_{ik}B^{kl} = \delta_i^l.$$

b) Demonstrate how using the reciprocal contravariant metric tensor g^{ik} and the covariant metric tensor g_{ik} you can form contravariant tensor from covariant tensors and vice versa.

We can introduce a contravariant metric tensor g^{ik} which is reciprocal to the covariant metric tensor g_{ik}:

$$g_{ik}g^{kl} = \delta_i^l.$$

With the help of the metric tensor and its reciprocal we can form contravariant tensor from covariant tensors and vice versa, for example:

$$A^i = g^{ik}A_k, \quad A_i = g_{ik}A^k.$$

c) Show that in an arbitrary non-inertial frame

$$g^{ik} = S^i_{(0)0}S^k_{(0)0} - S^i_{(0)1}S^k_{(0)1} - S^i_{(0)2}S^k_{(0)2} - S^i_{(0)3}S^k_{(0)3},$$

where $S^i_{(0)k}$ is the transformation matrix from locally inertial frame of reference (galilean frame) to this non-inertial frame.

We know that in the galilean frame of reference

$$g^{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv \eta^{ik} \equiv \mathrm{diag}(1, -1, -1, -1),$$

hence

$$g^{ik} = S^i_{(0)n}S^k_{(0)m}\eta^{lm} = S^i_{(0)0}S^k_{(0)0} - S^i_{(0)1}S^k_{(0)1} - S^i_{(0)2}S^k_{(0)2} - S^i_{(0)3}S^k_{(0)3}.$$

Q2.

a) Give a rigorous proof that the interval is a scalar.

Given that g_{ik} is a covariant tensor of the second rank and that

$$ds^2 = g_{ik}dx^i dx^k,$$

hence,

$$ds^2 = g_{ik}dx^i dx^k = (\tilde{S}_i^n \tilde{S}_k^m g'_{nm})(S_p^i dx'^p)(S_w^k dx'^w) = (\tilde{S}_i^n S_p^i)(\tilde{S}_k^m S_w^k)(g'_{nm}dx'^p dx'^w) =$$

$$= \delta_p^n \delta_w^m (g'_{nm}dx'^p dx'^w) = g'_{pw}dx'^p dx'^w = g'_{ik}dx'^i dx'^k = ds'^2,$$

thus

$$ds = ds'$$

which means that ds is a scalar.

b) Prove that the metric tensor is symmetric.

$$ds^2 = g_{ik}dx^i dx^k = \frac{1}{2}(g_{ik}dx^i dx^k + g_{ik}dx^i dx^k) = \frac{1}{2}(g_{ki}dx^k dx^i + g_{ik}dx^i dx^k) = \frac{1}{2}(g_{ki} + g_{ik})dx^i dx^k =$$

$$= \tilde{g}_{ik}dx^i dx^k,$$

where

$$\tilde{g}_{ik} = \frac{1}{2}(g_{ki} + g_{ik}),$$

which is obviously symmetric one. Then we just drop "~".

Q3.

Using lecture notes 3, write a short essay (1-2 pages) "Proper time and physical distances".

Proper time: the world line of an observer who uses some clock to measure the proper time, $d\tau$, between two infinitesimally close events in the same place in space is

$$dx^1 = dx^2 = dx^3.$$

Defining proper time exactly as in Special Relativity:

$$d\tau = \frac{ds}{c},$$

we have

$$ds^2 \equiv c^2 d\tau^2 = g_{ik} dx^i dx^k = g_{00}(dx^0)^2,$$

thus

$$d\tau = \frac{1}{c}\sqrt{g_{00}}dx^0.$$

For the proper time between any two events occurring at the same point in space we have

$$\tau = \frac{1}{c}\int \sqrt{g_{00}}dx^0.$$

Spatial distance: separating the space and time coordinates in ds we have

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + 2g_{0\alpha}dx^0 dx^\alpha + g_{00}(dx^0)^2.$$

To define dl we will use a light signal according to the following procedure: From some point B with spatial coordinates $x^\alpha + dx^\alpha$ a light signal emitted at the moment corresponding to time coordinate $x^0 + dx^{0(1)}$ propagates to a point A with spatial coordinates x^α and then after reflection at the moment corresponding to time coordinate x^0 the signal propagates back over the same path and is detected in the point B at the moment corresponding to time coordinate $x^0 + dx^{0(2)}$ as shown below. The interval between the events which belong to the same world line of light in Special and General Relativity is always equal to zero:

$$ds = 0.$$

Solving this equation with respect to dx^0 we find two roots:

$$dx^{0(1)} = \frac{1}{g_{00}}\left(-g_{0\alpha}dx^\alpha - \sqrt{(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00})dx^\alpha dx^\beta}\right)$$

$$dx^{0(2)} = \frac{1}{g_{00}}\left(-g_{0\alpha}dx^\alpha + \sqrt{(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00})dx^\alpha dx^\beta}\right)$$

$$dx^{0(2)} - dx^{0(1)} = \frac{2}{g_{00}}\sqrt{(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00})dx^\alpha dx^\beta}.$$

Then

$$dl = \frac{c}{2}d\tau = \frac{c}{2}\frac{\sqrt{g_{00}}}{c}(dx^{0(2)} - dx^{0(1)})$$

and finally

$$dl^2 = \gamma_{\alpha\beta}dx^\alpha dx^\beta, \quad \text{where} \quad \gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}.$$

Q4.

a) Show that all covariant derivatives of metric tensor are equal to zero.

$$DA_i = g_{ik} DA^k$$
$$DA_i = D(g_{ik} A^k) = g_{ik} DA^k + A^k Dg_{ik},$$

hence

$$g_{ik} DA^k = g_{ik} DA^k + A^k Dg_{ik},$$

which obviously means that

$$A^k Dg_{ik} = 0.$$

Taking into account that A^k is arbitrary vector, we conclude that

$$Dg_{ik} = 0.$$

Then taking into account that

$$Dg_{ik} = g_{ik;m} dx^m = 0$$

for arbitrary infinitesimally small vector dx^m we have

$$g_{ik;m} = 0.$$

b) Find the the relationship between the Cristoffel symbols and first partial derivative of the metric tensor.

Introducing useful notation

$$\Gamma_{k,\, il} = g_{km} \Gamma^m_{il},$$

we have

$$g_{ik;\, l} = \frac{\partial g_{ik}}{\partial x^l} - g_{mk} \Gamma^m_{il} - g_{im} \Gamma^m_{kl} = \frac{\partial g_{ik}}{\partial x^l} - \Gamma_{k,\, il} - \Gamma_{i,\, kl} = 0.$$

Permuting the indices i, k and l twice as

$$i \to k, \quad k \to l, \quad l \to i,$$

we have

$$\frac{\partial g_{ik}}{\partial x^l} = \Gamma_{k,\, il} + \Gamma_{i,\, kl}, \quad \frac{\partial g_{li}}{\partial x^k} = \Gamma_{i,\, kl} + \Gamma_{l,\, ik} \text{ and } -\frac{\partial g_{kl}}{\partial x^i} = -\Gamma_{l,\, ki} - \Gamma_{k,\, li}.$$

Taking into account that

$$\Gamma_{k,\, il} = \Gamma_{k,\, li},$$

after summation of these three equation we have

$$g_{ik,l} + g_{li,k} - g_{kl,i} = 2\Gamma_{i,\, kl},$$

and finally

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right).$$

Relativistic Astrophysics. 2007. Course Work 4. Solutions

Q1.

Write a brief essay (1-2 Pages) "Why invisible black holes can be seen in astronomical observations".

Black holes can be detected only by their gravitational interaction with matter and electromagnetic waves outside the event horizon. Although black holes cannot be detected directly, many observational studies have provided substantial evidence for black holes. Stellar mass black holes have masses that are equivalent to the masses of individual stars 415 times the mass of our Sun. Our Milky Way galaxy contains several probable stellar-mass black holes. These candidates are all members of X-ray binary systems in which the denser object draws matter from its partner via an accretion disk. A supermassive black hole is a black hole with a mass of an order of magnitude between 10^5 and 10^{10}. It is currently thought that almost all galaxies, including the Milky Way, contain supermassive black holes at their galactic centers. There is also evidence that two supermassive black holes can co-exist in the same galaxy for a certain amount of time.

Q2.

Explain briefly what is the main difference between the limit of stationarity and the event horizon of a black hole?

The Limit of stationarity (Static Limit): the interval ds for test particle in rest

$$dr = d\theta = d\phi = 0.$$

In this case

$$ds^2 = g_{00} dx^{0^2},$$

We can see that if

$$g_{00} = 0,$$

then

$$ds^2 = 0,$$

which means that the world line of particle in rest is the world line of light. Hence, at the surface

$$g_{00} = 0$$

no particle with finite rest mass can be in rest. For this reason this surface is called the limit of stationarity.

Event Horizon is a spherically symmetric surface

$$F(r) = \text{const}.$$

Its normal vector is defined as usually as

$$n_i = F_{,i} = \delta_i^1 \frac{dF}{dr}.$$

If at this surface

$$g^{11} = 0$$

then

$$g^{ik} n_i n_k = g^{11} n_1 n_1 = g^{11} \left(\frac{dF}{dr} \right)^2 = 0,$$

which means that n_i is a null vector and any particle with finite rest mass can not move outward the surface $g^{11} = 0$, thus this surface is the event horizon.

Q3.

Given that** n_i **is a covariant vector and

$$n^i = g^{ik} n_k.$$

Taking into account that** g_{ik} **and** g^{ik} **are reciprocal to each other, show that

$$g^{ik} n_i n_k = g_{ik} n^i n^k.$$

$$g^{ik} n_i n_k = g^{ik} g_{im} n^m g_{kp} n^p = \delta^k_m n^m g_{kp} n^p = g_{kp} n^m n^p = g_{ik} n^i n^k.$$

Q4.

Let the interval** ds **is given as

$$ds^2 = \left(1 - \frac{A}{x^1}\right)(dx^0)^2 - \left(1 - \frac{4A^2}{(x^1)^2}\right)^{-1}(dx^1)^2 - (dx^2)^2 - (dx^3)^2,$$

where** A **is a constant.

a) Find** g^{11}**.

$$g^{11} = \frac{g_{00} g_{22} g_{33}}{g_{00} g_{11} g_{22} g_{33}} = \frac{1}{g_{11}} = -\left(1 - \frac{4A^2}{(x^1)^2}\right).$$

b) Show that this interval corresponds to a space-time geometry with the limit of stationarity and the event horizon. Determine the position of both these surfaces.

The Limit of stationarity corresponds to

$$g_{00} = \left(1 - \frac{A}{x^1}\right) = 0,$$

hence

$$x^1 = A,$$

this surface is the limit of stationarity;

Event Horizon corresponds to a spherically symmetric surface

$$g^{11} = -\left(1 - \frac{4A^2}{(x^1)^2}\right) = 0,$$

hence

$$x^1 = 2A$$

is the event horizon.

Relativistic Astrophysics. 2007. Course Work 5. Solutions

Q1.

a) Discuss briefly what is the significant deference between the "Laplacian black hole" and the black hole in General Relativity.

In the case of "Laplacian black hole" a body with the velocity less than velocity of light, at the beginning moves outward and only after some time starts to move inward, while in the case of black hole in General Relativity motions outward are impossible, because the surface $r = r_g$ is the event horizon.

b) Explain why the surface $r = r_g$ in the Schwarzschild metric is the event horizon. Where the limit of stationarity is located?. Show that the surface $r = r_g$ is a null surface.

$$F(r) = r - r_g = 0$$

is obviously a spherically symmetric surface and its normal vector is

$$n_i = F_{,i} = \delta_i^1 \frac{dF}{dr} = \delta_i^1$$

At this surface

$$g^{11} = \frac{1}{g_{11}} = 1 - \frac{r_g}{r} = 1 - \frac{r_g}{r_g} = 0,$$

hence

$$g^{ik} n_i n_k = g^{11} n_1 n_1 = g^{11} = 0,$$

which means that n_i is a null vector and any particle with finite rest mass can not move outward the surface $g^{11} = 0$, thus this surface is the event horizon.

The limit of stationarity is located at the surface

$$g_{00} = 1 - \frac{r_g}{r} = 0,$$

which in the Schwarzschild metric is located also at the surface

$$r = r_g.$$

This surface is called null surface because the world line

$$r = r_g,\ d\theta = 0,\ d\phi = 0,$$

obviously belong to the surface

$$F(r) = r - r_g = 0$$

and corresponds to a world line of a photon moving outward with the speed of light.

Q2.

Consider a rotating black hole described by the Kerr metric.

a) What is meant by the event horizon, the "limit of stationarity" and the "ergosphere"? (Compare with the case of the Schwarzschild black hole).

The Kerr metric describing the gravitational field of rotating bodies has the following form:

$$ds^2 = (1 - \frac{r_g r}{\rho^2})dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_g r a^2}{\rho^2}\sin^2\theta)\sin^2\theta d\phi^2$$

$$+\frac{2r_g r a}{\rho^2}\sin^2\theta d\phi dt,$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - r_g r + a^2$, and $a = \frac{J}{mc}$, where J is the specific angular momentum.

For the Kerr metric $g_{00} = 0$ gives

$$1 - \frac{r_g r}{\rho^2} = 0,$$

thus

$$r^2 - r_g r + a^2\cos^2\theta = 0,$$

$$\Delta = r^2 - r_g r + a^2 = 0,$$

and

$$r_{st} = \frac{1}{2}(r_g \pm \sqrt{r_g^2 - 4a^2\cos^2\theta}) = \frac{r_g}{2} \pm \sqrt{(\frac{r_g}{2})^2 - a^2\cos^2\theta}.$$

The location of horizon in the Kerr metric: $g^{11} = 0$ ($g_{11} = \infty$) corresponds to

$$\Delta = r^2 - r_g r + a^2 = 0,$$

and

$$r = \frac{1}{2}(r_g \pm \sqrt{r_g^2 - 4a^2\cos^2\theta}) = \frac{r_g}{2} \pm \sqrt{(\frac{r_g}{2})^2 - a^2\cos^2\theta}.$$

$$r_{hor} = \frac{r_g}{2} \pm \sqrt{\left(\frac{r_g}{2}\right)^2 - a^2}.$$

One can see easily that

$$r_{st} \geq r_{hor},$$

for example,

$$r_{st} = r_{hor}, \text{ if } \theta = 0, \text{ or } \theta = \pi \text{ (at the poles)},$$

and

$$r_{st} = 2r_g > r_{hor}, \text{ if } \theta = \frac{\pi}{2} \text{ (at the equator)}.$$

The region between the limit of stationarity and the event horizon is called the "ergosphere".

In the Schwarzschild metric as was shown in the previous question

$$r_{hor} = r_{st},$$

which means that in this case the "ergosphere" does not exist.

b) Describe briefly the Penrose process of extraction of energy from a rotating black hole and explain why this mechanism does not contradict to the statement, that nothing can escape from within black hole.

By the Penrose mechanism it is possible to extract rotational energy of Kerr black hole. That extraction is made possible because the rotational energy of the black hole is located not inside the event horizon, but outside in a curl gravitational field. Such field is also called gravimagnetic field. All objects in the ergosphere are unavoidably dragged by the rotating spacetime. The Penrose mechanism: Some body enters into the "ergosphere" and decays then into two pieces. The momentum of the two pieces of matter can be arranged so that one piece escapes to infinity, whilst the other falls past the outer event horizon into the black hole. The escaping piece of matter can have greater mass-energy than the original infalling piece of matter.

Q3.

a) Give the definition of the Ricci tensor R_{ik} and prove that

$$R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}$$

By definition the Ricci tensor is

$$R_{ik} = g^{lm} R_{limk} = R^l_{ilk},$$

where the curvature Riemann tensor is defined by

$$A_{i;\,k;\,l} - A_{i;\,l;\,k} = A_m R^m_{ikl}.$$

By straightforward calculations

$$\begin{gathered}
A_{i;k;l} - A_{i;l;k} = \\
A_{i;k,l} - \Gamma^m_{li} A_{m;k} - \Gamma^m_{lk} A_{i;m} - \\
-A_{i;l,k} + \Gamma^m_{ki} A_{m;l} + \Gamma^m_{kl} A_{i;m} = \\
\left(A_{i,k} - \Gamma^m_{ik} A_m\right)_{,l} - \Gamma^m_{li}\left(A_{m,k} - \Gamma^n_{mk} A_n\right) - \\
-\left(A_{i,l} - \Gamma^m_{il} A_m\right)_{,k} + \Gamma^m_{ki}\left(A_{m,l} - \Gamma^n_{ml} A_n\right) = \\
A_{i,k,l} - A_{i,l,k} - \Gamma^m_{ik} A_{m,l} - \Gamma^m_{il} A_{m,k} - \Gamma^m_{kl} A_{i,m} + \Gamma^m_{il} A_{m,k} + \Gamma^m_{ik} A_{m,l} + \Gamma^m_{lk} A_{i,m} - \\
-\Gamma^m_{ik,l} A_m + \Gamma^m_{il}\Gamma^p_{mk} A_p + \Gamma^m_{kl}\Gamma^p_{im} A_p + \\
+\Gamma^m_{ik,l} A_m - \Gamma^m_{ik}\Gamma^p_{ml} A_p - \Gamma^m_{lk}\Gamma^p_{im} A_p = \\
= A_m\left(-\Gamma^m_{ik,l} + \Gamma^p_{il}\Gamma^m_{pk} + \Gamma^p_{kl}\Gamma^m_{ip} + \Gamma^m_{il,k} - \Gamma^p_{ik}\Gamma^m_{pl} - \Gamma^p_{lk}\Gamma^m_{ip}\right) = \\
= A_m\left(-\Gamma^m_{ik,l} + \Gamma^p_{il}\Gamma^m_{pk} + \Gamma^m_{il,k} - \Gamma^p_{ik}\Gamma^m_{pl}\right),
\end{gathered}$$

hence

$$R^m_{ikl} = \Gamma^m_{il,k} - \Gamma^m_{ik,l} + \Gamma^p_{il}\Gamma^m_{pk} - \Gamma^p_{ik}\Gamma^m_{pl},$$

and replacing k by l and l by k and then just putting $m = l$ we finally obtain

$$R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}.$$

b) Starting from the Einstein equations in the form

$$R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi G}{c^4} T_{ik},$$

where G is the gravitational constant, prove that

$$T^i_k = \frac{c^4}{8\pi G} \left(R^i_k - \frac{1}{2} \delta^i_k R \right).$$

Contracting with g^{ik}, we have the Einstein equations in mixed form

$$R^i_k = \frac{8\pi G}{c^4} (T^i_k - \frac{1}{2} \delta^i_k T).$$

$$R = g^{ik} R_{ik} = \frac{8\pi G}{c^4} \left(g^{ik} T_{ik} - \frac{1}{2} g^{ik} g_{ik} T \right) = \frac{8\pi G}{c^4} \left(T^i_i - \frac{1}{2} \delta^i_i T \right) = \frac{8\pi G}{c^4} \left(T - \frac{1}{2} 4 \right) = -\frac{8\pi G}{c^4} T.$$

Thus

$$T = -\frac{c^4}{8\pi G} R.$$

Thus

$$T_{ik} = \frac{c^4}{8\pi G} \left(R_{ik} - \frac{1}{2} g_{ik} R \right),$$

then in mixed form we have

$$T^i_k = \frac{c^4}{8\pi G} \left(R^i_k - \frac{1}{2} \delta^i_k R \right).$$

c) What can you say about the nature of gravitational field, for which $R_{ik} = 0$, while R_{ikln} is not equal to zero?

This situation corresponds to gravitational fields (for example, gravitational waves), when the space-time is curved, but matter is absent (empty space-time).

Relativistic Astrophysics. 2007. Course Work 6. Solutions

Q1.

A binary system consists of a invisible compact object of mass M_x and a visible star of mass M. The period of the orbit is T, the angle between the normal to the plane of the orbit to the line of sight of the observer is i and projection of orbital velocity of the visible star on the line of sight is v.

a) Which of values M_x, M, T, i and v can be determined from observations and how?

The observable values are T and v; The period T is measured by clocks, the velocity v is obtained from spectroscopic measurements based on Doppler effect.

b) Using Newtonian theory show that

$$\frac{(M_x \sin i)^3}{(M_x+M)^2} = \frac{v^3 T}{2\pi G}.$$

We need solve the following system if equations:

$$rM = r_x M_x,$$

$$\omega^2 r_x = GM(r_x+r)^{-2},$$

$$\omega^2 r = GM_x(r_x+r)^{-2},$$

$$v = \omega r \sin i.$$

Summing the second with the third, we have

$$\omega^2(r_x+r) = GM(r_x+r)^{-2},$$

and

$$r_x + r = \left[\frac{G(M+M_x)}{\omega^2}\right]^{1/3},$$

$$r_x = r\frac{M}{M_x},$$

$$r(1+\frac{M}{M_x}) = \left[\frac{G(M+M_x)}{\omega^2}\right]^{1/3},$$

$$v = \omega r \sin i = \omega \sin \frac{M_x}{M+M_x}\left[\frac{G(M+M_x)}{\omega^2}\right]^{1/3},$$

$$= (G\omega)^{1/3}\sin i \frac{M_x}{(M+M_x)^{2/3}},$$

$$\frac{v^3}{G\omega} = \frac{v^3 T}{2\pi G} = \frac{M_x^3 \sin^3 i}{(M_x+M)^2}.$$

c) Assume that observations of a binary system give strong evidence, that the visible star is periodically eclipsed by the invisible object (eclipsed binary). What can you say in this case about the orientation of the binary system?

In the case we can say, that the line of sight is very close to the orbital plane, which means that $\sin i \approx 1$.

d) Imagine that you obtained several sets of observations for eclipsed binaries. Results of these observations are summarized below:

Object N	1	2	3	4	5
Velocity in km/s	200	300	500	1000	2000
Period in min	5	10	20	50	100

Assume that any invisible compact object is a black hole, if its mass exceeds $3M_\odot$. Assume also that the mass of all visible stars in your binaries have masses between $M_\odot$ and $5M_\odot$. Which binary system contains, may contain and does not contain a black hole?

For all these objects $\sin i \approx 1$.
Introducing

$$m_x = M_x/M_\odot,$$
$$m = M/M_\odot$$

and

$$f = \frac{m_x^3}{(m_x+m)^2}$$

we have:

for $m_x = 3$ and $m = 1$

$$f = \frac{27}{16} \approx 1.7,$$

for $m_x = 3$ and $m = 5$

$$f = \frac{27}{64} \approx 0.42.$$

Thus:

if $f < 0.42$ there is no black hole,
if $0.42 < f < 1.7$ there may be a black hole,
if $1.7 < f$ there is a black hole.

Taking into account that according to Q1(b)

$$f = \left(\frac{v^3 T}{2\pi G M_\odot}\right) = \left(\frac{v}{100\text{km s}^{-1}}\right)^3 \left(\frac{T}{5\text{min}}\right) \left(\frac{10^6 \cdot 300}{\pi c^2 \frac{2GM_\odot}{c^2}}\right) \text{km}^3\text{s}^{-2} =$$

$$= \left(\frac{v}{100\text{km s}^{-1}}\right)^3 \left(\frac{T}{5\text{ min}}\right) \left(\frac{10^{-2}}{3\pi r_{g\odot}}\right) \text{km} = 3,5 \cdot 10^{-4} \left(\frac{v}{100\text{km s}^{-1}}\right)^3 \left(\frac{T}{5\text{ min}}\right).$$

Object 1: $f \approx 0.0028$.
Object 2: $f \approx 0.019$.
Object 3: $f \approx 0.177$.
Object 4: $f \approx 3.54$.
Object 5: $f \approx 56.62$.

Thus, the final answer is:
the objects N 4 and 5 contain black holes, the object N 1 - 3 do not contain black holes.

Q2.

The "effective potential energy" is defined as

$$U(r) = mc^2 \left(1 - \frac{r_g}{r}\right)^{1/2} \left(1 + \frac{L^2}{m^2c^2r^2}\right)^{1/2},$$

where L is the angular momentum and m is the mass of a particle, moving around Schwarzschild black hole.

a) What is the physical meaning of the "effective potential energy"? Explain how using U to find stable and unstable circular orbits.

The effective potential energy includes potential energy and that part of kinetic energy, which is related with non-radial, angular motion. Points at which $E = U$, (E is the conservative total energy) correspond to turning points, where $dr/dt = 0$.

$$U = E, \;\; U_r^{'} = 0,$$

corresponds to the circular orbit, stable, if $U_{rr}^{''} > 0$, and unstable, if $U_{rr}^{''} < 0$.

b) Using the Hamilton-Jacobi equation, show that the energy of a particle moving along circular orbit depends on the radius of the orbit as follows

$$E(r) = \sqrt{2}mc^2 \frac{(r - r_g)}{(2 - 3r_g)^{1/2}\, r^{1/2}}.$$

Introducing $x = r_g/r$, we have $U_r^{'} = 0$ corresponds $U_x^{'} = 0$, so

$$[(1 - x)(1 + \alpha x^2]_x^{'} = 0,$$

where

$$\alpha = \frac{L^2}{m^2c^2r_g^2},$$

$$-1 - 3\alpha x^2 + 2\alpha x = 0,$$

and

$$\alpha = \frac{1}{x(2 - 3x)}.$$

Then

$$\frac{E^2}{m^2c^4} = (1 - x)(1 + \frac{x}{2 - 3x}) = \frac{2(1 - x)^2}{3 - 3x},$$

and finally

$$E = \frac{\sqrt{2}mc^2(1 - r_g/r)}{(2 - 3r_g/r)^{1/2}} = \frac{\sqrt{2}mc^2(r - r_g)}{(2r - 3r_g)^{1/2}r^{1/2}}.$$

c) Determine the radius of the last circular orbit. What fraction of the initial energy will be released by the particle when it reaches the last circular orbit?

The last circular orbit corresponds the following system of equations: $E = U$, $U^{'} = 0$, $U^{''} = 0$.

$$0 = U^{''} \sim 2\alpha(1 - 3x),$$

so $x = 1/3$, which corresponds to $r = 3r_g$.

$$\frac{E^2}{m^2c^4} = (1 - 1/3)(1 + 3/3^2) = 8/9,$$

and

$$E_{lo} = mc^2\frac{2\sqrt{2}}{3}.$$

Fraction of energy:

$$f = \frac{E_\infty - E_{lo}}{E_\infty} = 1 - \frac{2\sqrt{2}}{3} = 0.057$$

Q3.

The quadrupole formula for the metric perturbation associated with gravitational waves is given by

$$h_{\alpha\beta} = -\frac{2G}{3c^4R}\frac{d^2D_{\alpha\beta}}{dt^2}(t - R/c),$$

where R is the distance to the source of the gravitational waves and

$$D_{\alpha\beta} = \int (3x_\alpha x_\beta - r^2\delta_{\alpha\beta})dM$$

is the quadrupole tensor of the source. Consider a mass m moving along circular orbit around the black hole of mass M, assuming that $m \ll M$.

a) Show that all the amplitudes $h_{\alpha\beta}$ of gravitational wave, emitted by such system, are periodic functions of time with $\omega = 2\omega_0$, where $\omega_0 = 2\pi/T$, and T is the orbital period.

$$x_1 = r\cos\omega_0 t,$$

$$x_2 = r\sin\omega_0 t,$$

$$D_{11} = mr_c^2(3\cos^2\omega_0 t - 1) = \frac{1}{2}mr^2(1 + 3\cos 2\omega_0 t),$$

$$D_{22} = mr_c^2(3\sin^2\omega_0 t - 1) = \frac{1}{2}mr^2(1 - 3\cos 2\omega_0 t),$$

$$D_{12} = \frac{3}{2}mr_c^2\sin 2\omega_0 t,$$

then

$$h_{11} = -\frac{2Gmr^2}{3c^4R}\frac{3}{2}(2\omega_0)^2\cos 2\omega_0 t = \frac{4\omega_0^2Gmr^2}{c^4R}\cos 2\omega_0,$$

$$h_{22} = \frac{2Gmr^2}{3c^4R}\frac{3}{2}(2\omega_0)^2\cos 2\omega_0 t = -\frac{4\omega_0^2Gmr^2}{c^4R}\sin 2\omega_0,$$

$$h_{12} = \frac{2Gmr^2}{3c^4R}\frac{3}{2}(2\omega_0)^2\sin 2\omega_0 t = \frac{4\omega_0^2Gmr^2}{c^4R}\sin 2\omega_0,$$

it is clear, that

$$\omega = 2\omega_0.$$

b) Show that, to an order of magnitude (omitting the indices α and β)

$$h \approx \frac{r_g}{R}\left(\frac{R_g \omega}{c}\right)^{2/3},$$

where r_g is the gravitational radius of the mass m and R_g is the gravitational radius of the black hole.

From

$$r\omega_0^2 = \frac{GM}{r^2},$$

we have

$$\frac{1}{r^3} = \frac{\omega_0^2}{GM},$$

and finally

$$r_c^{-1} = (4GM)^{-1/3}\omega^{2/3}.$$

Thus

$$h \approx \frac{4\omega_0^2 G m r^2}{c^4 R} = \frac{r_g R_g}{rR} \approx \frac{r_g}{R}\left(\frac{R_g \omega}{c}\right)^{2/3}.$$

c) The future LISA mission will be able to detect gravitational waves with $h > 10^{-23}$, if $10^{-4} Hz < \omega < 3 \cdot 10^{-3} Hz$. From what distance will it be possible to detect gravitational radiation from the binary system, containing the black hole of mass $m = 3M_\odot$, moving along a circular orbit with radius $r = 10^4 R_g$ around the massive black hole of mass $M = 10^3 M_\odot$?

$$\omega_0^2 = \frac{GM}{r^3} = \frac{+}{c^2}2\frac{2GM}{c^2 r^3} = c^2\frac{R_g}{2r^3},$$

hence,

$$\omega_0 = c\sqrt{\frac{R_g}{2r^3}} = c\sqrt{\frac{R_g}{2 \cdot 10^{12} R_g^3}} = \frac{10^{-6}c}{\sqrt{2}R_g} = \frac{10^{-4}Hz}{\sqrt{2}},$$

thus

$$\omega = 2\omega_0 = \sqrt{2}10^{-4}Hz \geq 10^{-4}Hz,$$

which means that the radiation is within LISA frequency range.

$$h = \frac{3 \cdot 10^5}{3 \cdot 10^{18}}\left(\frac{3 \cdot 10^5 \cdot 10^{-4}}{3 \cdot 10^{10}}\right)^{2/3}(\frac{m}{M.})(\frac{R}{1pc})^{-1}(\frac{M}{M.})^{2/3}(\frac{\omega}{10^{-4}Hz})^{2/3}$$
$$\approx 10^{-19}(\frac{m}{M.})(\frac{R}{1\ pc})^{-1}(\frac{M}{M.})^{2/3}(\frac{\omega}{10^{-4}Hz})^{2/3}.$$

Then

$$h = \frac{3 \cdot 10^5 cm}{R}(\frac{3 \cdot 10^5 \cdot 10^3 \cdot 1.4 \cdot 10^{-4} s^{-1} cm}{3 \cdot 10^{10}})^{2/3} > 10^{-23},$$

if

$$R < 3 \cdot 10^{23} \cdot 10^5 cm \cdot 10^{-4} \approx 1Mpc.$$