Professional Skills for Physicists: II

Problem Solving: Section D

Longer questions and 2007 test paper

LQ 1:

(a) The Fermi energy E_F of the free electron gas in a metal depends on the following: n_e, the number density of electrons; m_e, the mass of the electron, and $\hbar$ (Planck's constant divided by 2π). It accordingly takes the form:-

$$E_F = C n_e^u m_e^v \hbar^w$$

where the numerical constant, C, is 4.8 in SI units. Use dimensional arguments to find the required values of the numerical constants, u, v and w.

(b) The density of electrons in metallic sodium is 2.7×10^{28} m^{-3}. Estimate the Fermi energy of the electrons in *Joules* and in *electron volts*.

LQ 2:

(a) By considering the electrostatic force between two point charges e, or otherwise, show that the dimensions of (e^2/ϵ_o) are required to be ML^3T^{-2}. (As usual ϵ_o is the permittivity of free space.)

(b) The binding energy of the lowest state of the hydrogen atom according to the Bohr model is given by the formula:-

$$E_1 = kh^\alpha m^\beta (e^2/\epsilon_o)^\gamma,$$

where k is a numerical constant and h and m are respectively Planck's constant and the mass of the electron. Use dimensional analysis to obtain numerical values for the powers α, β and γ.

(c) Given that $k = \frac{1}{8}$, obtain an estimate of the energy E_1, in electron volts.

(d) Write down an estimate – however rough – of the radius of the hydrogen atom (the radius of the lowest Bohr orbit). Use it to find an approximate value for the density of hydrogen assuming that the atoms are packed with the mean atomic spacing equal to twice their radii. Compare your answer with

(i) the density of hydrogen gas at STP,

(ii) the mean density of the planet Neptune (mass 10^{26} kg, radius 25000 km, and composed mainly of hydrogen).

What can you deduce from this about the physical state of Neptune's interior?

LQ 3:

Applying simple physical ideas as appropriate, estimate the following quantities:

(a) The land area needed for a terrestrial 1000 MW solar power station, given that the Sun radiates at a power of 4×10^{26} W,

(b) The power required to keep the air in a house warm in winter, if draughts cause the complete replacement each hour of the warm air by cold air from outside.

LQ 4: A tougher example

(a) A new system of units is devised in which unit length remains 1 metre, but the units of time and mass are so chosen that the speed of light c and the gravitational constant G are both of magnitude unity. What will be the new unit of mass in kg?

(b) Particle physicists find it convenient to use a different simplified system of units in which the speed of light, only, is taken to be unity. This enables the use of the following expression for the total energy E of a particle with rest mass m_o:

$$E^2 = p^2 + m_o^2,$$

where p is the particle momentum. If an electron is claimed to have an energy of 1.5 MeV, what will its momentum be in MeV/c (to within about 10 percent)?

LQ 5:

There is a physical effect such that electromagnetic waves are shielded from penetrating electrically-conducting material beyond a representative depth, δ, with the consequence that wave amplitude declines inwards according to:

$$A(x) = A(0)\exp(-x/\delta),$$

where $x = 0$ at the surface of the conductor and x increases with depth into the conductor, and δ is the 'skin depth' given by:

$$\delta = \frac{1}{\sqrt{\pi\sigma\nu\mu_r\mu_o}}.$$

In this expression ν is the frequency of incident radiation, σ is the conductivity of the material and μ_r is its relative permeability.

(a) Show that δ has the required dimensions of length.

(b) Estimate the skin depth of copper when illuminated by visible light. At room temperature the conductivity of copper is approximately $6{\times}10^7$ Ohm^{-1} m^{-1}. Assume its relative permeability to be 1.

(c) One result of this 'skin effect' is that alternating current carried by a metal wire is excluded from all but a thin outer skin of the wire, of a thickness comparable with δ. Consider another – how, qualitatively, might this skin effect also explain the visible appearance of a metal like copper or aluminium?

LQ 6:

(a) Carry out an estimation to confirm that the mass density of air at normal temperature and pressure (i.e. at STP) is about 1 kg m^{-3}.

(b) Giving some justification for the value you choose, estimate the mean wind speed that would be measured at the top of the Blackett Laboratory.

(c) If a request to build a windmill of blade area 1 m^2 were not blocked by local residents, and it was built, estimate the mean power such a windmill would generate.

(d) An adult cyclist averages 5 m s^{-1} in London, and has to stop every 200 m, typically. Estimate the mean energy per kilometre that the cyclist will dissipate
(i) due to wind resistance, (ii) due to breaking to stop.

(e) Our cyclist, who commutes into Blackett daily, gets lazy and decides to buy an electric bike. But she is so racked by environmental guilt, that she decides to charge it up only through the Blackett roof windmill. How close to college must the cyclist live for this to be a viable commuting solution?

LQ 7: *[This is adapted from part of a previously set question in a level 2 option paper. The numerical estimation is the challenge, not concept – it really doesn't matter at all in the manipulation and evaluation that the electrons within the white dwarf find themselves in a 'degenerate gas'. Just pick out and follow the instructions.]*

(i) A white dwarf may be treated as a uniformly dense sphere of mass density ρ comprising nuclei of C and O, permeated by a degenerate gas of free electrons. Use the assumption that the number of electrons is equal to half the number of nucleons, to relate the electron number density n_e to the total mass density (ρ).

(ii) For such a uniformly dense sphere of radius R and mass M, the central pressure is $P_c = 3GM^2/(8\pi R^4)$. Equate this to the pressure of a degenerate non-relativistic free-electron gas, $P = \hbar^2(3\pi^2)^{2/3}n_e^{5/3}/(5m_e)$, and transform your expression into a relation between radius and mass for a white dwarf.

(iii) Derive an approximate value for the radius of a white dwarf of mass $1\,M_\odot$, expressing your answer in units of the solar radius, $R_\odot$.

Astronomical observations imply typical white dwarf radii of close to 0.01$R_\odot$. Is the difference between your numerical estimate and this finding acceptable?

Problem Solving Test

Friday, April 27, 2007: 10.00-11.30

Instructions:

Attempt question 1 and one other (one of questions 2, 3 or 4). Write the answer to each question in a separate answerbook.

Each question is worth 50 % of the total marks available, if it is answered *legibly*, *with clear reasoning*, and *correctly*. In the right-hand margin of each question you will find an indicative marking scheme.

No electronic calculators are to be used.

Please ensure the test paper supplied to you includes all 4 questions, and also the list of common physical and astronomical constants (3 pieces of paper, altogether).

Question 1. (Compulsory)

When two flat plates are brought very close together in a vacuum, there is an attractive force between them that is not due to gravity or any charge on them. This surprising phenomenon is known as the Casimir Effect, and the force is due to the "vacuum energy", a construct that is explained by quantum field theory.

The force between the two plates per unit area – in effect, a pressure P – is known to depend on the speed of light, Planck's constant, and the distance d between the two plates.

(a) Use the technique of dimensional analysis to show that $P = \kappa hc/d^n$, where κ is a numerical constant of proportionality, and n is a positive integer. [5]

(b) Estimate the value of κ, to within a few percent, given that the Casimir Effect produces the equivalent of 1 standard atmosphere of pressure (1.01×10^5 Nm^{-2}) when the plates are separated by 10.6 nm. [4]

(c) By how much, as a rough percentage, would the value for the constant κ change if the supplied value for d had been rounded down to 10 nm? [4]

(d) Estimate the pressure when the plates are separated by 1 mm. [3]

(e) It was asserted above that the attractive force cannot be attributed to Newtonian gravity. Comment briefly on how, by analysis and calculation, you could demonstrate this assertion is correct, for a circumstance like that envisaged in part (b). What difficulties stand in the way of doing this with anything better than order of magnitude accuracy? Do not perform any calculations. [4]

Question 2.

The moment of inertia, I, of a solid about a specified rotation axis is obtained by integrating over the mass moment distribution about the rotation axis. Seen in dimensional terms, I is equivalent to the mass M of the object multiplied by the square of an appropriately-weighted radius. This "appropriately-weighted radius" is what is known as the *radius of gyration*, k, which allows the moment of inertia for any solid to be written as $I = Mk^2$. The more closely the mass of an object is packed around the axis of rotation, the smaller k will be (compared to the object's actual size). Use this way of expressing the moment of inertia in considering the following problem:

(a) Consider a ball rolling down a slope such that it does not slip. Establish an equation relating the loss of gravitational energy to the translational and rotational kinetic energies gained. Your equation should involve h the vertical height lost by the ball, the ball's translational speed v, its radius r and radius of gyration k. [5]

(b) Two balls are let go at the top of the same slope at the same time. They are of the same mass but one is hollowed out, while the other is solid and uniform. Using your energy equation, work out whether the two balls – which roll down without slipping – reach the bottom of the slope at the same time. If one of them does reach the bottom first, which will it be? [6]

(c) A tennis ball and a cricket ball are about the same size, although a tennis ball is rough-textured and essentially hollow, while a cricket ball is smoother, more massive and close to being a uniform solid. These are allowed to roll down a slope such that, when they reach the bottom, they are 1 metre below where they started from at the top. Estimate their respective radii of gyration and translational speeds at the foot of the slope. [6]

(d) In the above, it has not been necessary to specify the angle of the slope. Why not? What would it be about the physics of balls rolling down slopes that would make this angle relevant? Give your opinion on this in two or three sentences, drawing on the example you have just done. [3]

Question 3.

The London Underground has a problem with waste heat: from data gathered in 2005-6, it is known that London Underground uses close to 500 GW-hours in one year, and that about half of it is currently released (i.e. wasted) as heat. The tube network is just over 400 km long, with roughly half of it actually under ground, and there are about 275 stations.

(a) Using the above data, estimate the rate of heat wastage into the tunnels of the network. [4]

(b) Transport for London wants to limit the difference between the ambient air temperature and the temperature of under-ground air to 5 K (being anxious to avoid the scandal of dead tube passengers on a globally-warmed summer's day). Given the rate of heat wastage you just calculated, what would be the maximum timescale, Δt, within which it would be necessary to completely replace the air in the underground tunnels to keep within this limit. [6]

(c) Currently, the air in the tunnels is exchanged by using platform fans that are fed tunnel air delivered by the movement of trains through the tunnels – the so-called piston effect.

It is known from modelling cars in a tunnel that the volume of air exchanged per unit time can be written:

$$L = \frac{A_t v}{\sqrt{\frac{\xi A_t}{N c A_v}} + 1},$$

where A_t and A_v are the cross-sections of the tunnel and vehicles passing through, respectively. The average vehicle speed is v. The coefficients, ξ and c, are hydrodynamic resistance and drag coefficients, while N is the number of vehicles in the tunnel.

(i) Is the equation dimensionally sound? What can you say, dimensionally, about the ratio of coefficients ξ/c in the denominator? [3]

(ii) Assuming that the value of ξ/c is of order unity, perform a rough numerical calculation to see how useful the piston effect is in replacing London Underground tunnel air. [5]

(d) Comment on your result: does this amount to a workable system, and how would you cool the tunnel and platform network more efficiently if this is or were to become necessary. [2]

Question 4.

(a) Show that what is known as the *solar constant*, i.e. the radiant energy from the Sun passing through unit area in unit time into the Earth's atmosphere, is about 1.4 kW m^{-2}. [4]

(b) Estimate the number of photons from the Sun incident per unit time, per unit area, at the surface of the Earth. You may assume that all of the Sun's emission takes place in the green part of the spectrum ($\lambda \sim 500$ nm). Also note the atmosphere removes two-thirds of the incident solar radiation before it reaches the Earth's surface. [4]

(c) By considering dimensions or otherwise, show that momentum per unit time per unit area is equivalent to both pressure and an energy density. [3]

(d) A single photon carries a momentum of $h\nu/c$ where ν is the frequency of radiation, h is Planck's constant and c is the speed of light. Evaluate the maximum pressure due to sunlight at the Earth's surface. [5]

(e) The world's most powerful laser can currently generate 1$\times$10^{15} W of radiant power. If this laser is focused to a small circular spot of radius $r = 10$ μm, derive an order of magnitude estimate for the pressure produced at the target surface. Express your answer in atmospheres (1 atmosphere $\simeq$1.0$\times$10^5 N m^{-2}). [4]

Common physical and astronomical constants

Speed of light in a vacuum, c:	3×10^8 m s^{-1}
Planck's constant, h:	6.6×10^{-34} J s
Universal gravitational constant, G:	6.7×10^{-11} N m^2 kg^{-2}
Permittivity of the vacuum, ϵ_o:	8.8×10^{-12} F m^{-1}
Permeability of the vacuum, μ_o:	$4\pi \times 10^{-7}$ H m^{-1} (or N A^{-2})
Electron rest mass, m_e:	9.1×10^{-31} kg
Proton rest mass, m_p:	1.7×10^{-27} kg
Electron charge, e:	1.6×10^{-19} C
Boltzmann's constant, k:	1.4×10^{-23} J K^{-1}
Stefan's constant, σ:	5.7×10^{-8} J m^{-2} s^{-1} K^{-4}
Avogadro's number, N_A:	6.0×10^{23} mol^{-1}
Molar gas constant, R:	8.3 J K^{-1} mol^{-1}
Volume occupied by 1 mole of gas at STP:	22.4 litres
Mass density of water:	1 kg litre^{-1} (Note: 1 litre = 0.001 m^3)
Mass of the Sun, $M_\odot$:	2×10^{30} kg
Radius of the Sun, $R_\odot$:	7×10^8 m
Luminosity of the Sun, $L_\odot$	3.8×10^{26} W
Mass of the Earth, M_E:	6×10^{24} kg
Radius of the Earth, R_E:	6400 km
Mean radius of Earth's orbit around Sun:	1.5×10^{11} m
Acceleration due to gravity on Earth's surface, g:	9.8 m s^{-2}
Length of the tropical year:	3.2×10^7 s