# Professional Skills for Physicists: II

## Problem Solving: Section B

### Numerical estimation and approximate methods

**January – March 2007**

## 9 Arithmetic without a calculator - the skill of estimation

All exercises in this hand-out should be attempted without a calculator. There are advantages in being able to do sums without a calculator:

- You can work out a rough answer whenever an idea comes into your head
- You keep a sense of the magnitude of numbers and effects and thereby better connect the maths with the physics
- It gives you a way of checking an answer arrived at with a calculator

> **Exercise 9.1:** What is the energy (in Joules) of a very high energy cosmic ray proton arriving at earth with a Lorentz factor of $3 \times 10^{11}$? Making your own estimate of the mass and velocity of a tennis ball, compare this with the energy of a tennis ball served by a top player.

The answer to the first part of exercise 9.1 could be calculated with pen and paper by exactly multiplying the Lorentz factor of $\gamma = 3 \times 10^{11}$ by the proton mass $m_p = 1.67 \times 10^{-27}\text{kg}$ by the speed of light ($c = 3 \times 10^8$) squared. Alternatively, you could recognise that (i) $1.67 \approx 5/3$, so $\gamma m_p \approx 5 \times 10^{-16}$, and (ii) $3 \approx \sqrt{10}$, so $c^2 \approx 10^{17}$. Putting these together gives $\gamma m_p c^2 \approx 50$ J. A similar estimate of the energy of a tennis ball gives a value that is not too different. The art of making estimates without using a calculator is to accept errors of the order of 20%, for example, in saying that $c^2 = (3 \times 10^8)^2 \approx 10^{17}$.

Useful tricks are

- $\pi \approx 3$, and $\pi^2 \approx 10$ to a few % accuracy
- Expressing numbers such as 1.67 as a fraction, 5/3
- Remembering the powers of 2, and also useful proxies for them (such as $2^6 \approx 200/3$ and $2^{10} \approx 1000$)
- Remembering some useful square roots, including $\sqrt{2} \approx 1.4$ and $\sqrt{3} \approx 1.7 \approx 5/3$, and cube roots such as $10^{1/3} \approx 2$
- Remembering that e=exp(1)$\simeq$2.7, ln(10) $\simeq$ 2.3 (or $\log_{10}$e$\simeq$ 0.43). These are helpful when faced with exponentials raised to extreme powers.

As with the tricks of dimensional analysis, it helps to look for useful combinations of numbers. This helped in exercise 9.1 in which it helped to group $\gamma$ and $m_p$ together. In similar vein, it is a good habit to group all the powers of 10 together and calculate them separately from the other numbers. Numerical estimation that gets it right is a crucial skill to develop: you need to establish your own preferred simplification tricks, and practise them!

**Worked example**

As an example, the energy of the $n$th energy level in the hydrogen atom is

$$E_n = \frac{m_e e^4}{8\epsilon_0^2 h^2 n^2} = \frac{9 \times 10^{-31} \times (1.6 \times 10^{-19})^4}{8 \times (8.8 \times 10^{-12})^2 \times (6.6 \times 10^{-34})^2 \times n^2} \text{ J}$$

The calculation can proceed in the following steps:

1) $1.6 \times 10^{-19} = 2^4 \times 10^{-20}$, giving $(1.6 \times 10^{-19})^4 = 2^{16} \times 10^{-80} = 2^{10} \times 2^6 \times 10^{-80} = 1000 \times 200/3 \times 10^{-80} = 2/3 \times 10^{-75}$.
2) Since 8.8 is close to 9 and $9^2 \approx 80$, $(8.8 \times 10^{-12})^2 \approx 8 \times 10^{-23}$.
3) Since $6.6 \approx 20/3$, $(6.6 \times 10^{-34})^2 \approx (4/9) \times 10^{-66}$.

Collecting the powers of 10 together gives

$$E_n \approx \frac{9 \times (2/3)}{8 \times 8 \times (4/9)} \times \frac{10^{-19}}{n^2} \approx \frac{540}{256} \times \frac{10^{-18}}{n^2} \approx (\frac{512}{256} + \frac{28}{256}) \times \frac{10^{-18}}{n^2}$$
$$\approx 2.1 \times 10^{-18} n^{-2} \text{ J}$$

The exact result is $2.2 \times 10^{-18} n^{-2}$J, so the approximate calculation is really quite accurate.

There are many ways this calculation might be performed. If less accuracy is required, there are short-cuts that could make the calculation even easier, as the following example shows:

**Re-worked example**

As above,

$$E_n = \frac{m_e e^4}{8\epsilon_0^2 h^2 n^2} = \frac{9 \times 10^{-31} \times (1.6 \times 10^{-19})^4}{8 \times (8.8 \times 10^{-12})^2 \times (6.6 \times 10^{-34})^2 \times n^2} \text{ J}$$

If less accuracy is required, the calculation can proceed in the following steps:

1) $(1.6 \times 10^{-19})^4 = (1.6)^4 \times 10^{-76} \approx (\sqrt{3})^4 \times 10^{-76} \approx 10^{-75}$,

2) $9 \times 10^{-31} \approx 10^{-30}$ and $8 \approx 10$

3) $(8.8 \times 10^{-12})^2 \approx (10^{-11})^2 \approx 10^{-22}$.

4) $(6.6 \times 10^{-34})^2 \approx (4/9) \times 10^{-66} \approx 4 \times 10^{-67}$.

Putting these together gives

$$E_n \approx \frac{10^{-30} \times 10^{-75}}{10 \times 10^{-22} \times 4 \times 10^{-66} n^2} \approx 2.5 \times 10^{-18} n^{-2} \text{ J}$$

The less accurate calculation still does surprisingly well, and for many purposes the accuracy is quite sufficient.

**Exercise 9.2:** Without a calculator, evaluate the following:

(i) The Bohr radius

$$a_0 = \frac{\epsilon_0 h^2}{\pi m_e e^2}$$

(ii) The Thomson scattering cross-section

$$\sigma = \frac{8\pi}{3} (\frac{e^2}{4\pi\epsilon_0 m_e c^2})^2$$

(iii) The astronomical unit of length, the parsec. This is defined as the distance from the solar system that an object has to be in order that its parallax (half its maximum angular displacement against a background of 'fixed' very distant objects seen during the course of a year) is one second of arc.

(iv) The Boltzmann ratio at (a) $T = 100000$ K and (b) room temperature, giving the fraction of H atoms excited to the $n = 2$ level relative to those still in the ground state ($n = 1$):

$$\frac{N_2}{N_1} = 4 \exp(\frac{-\chi_{1,2}}{kT})$$

Here $\chi_{1,2}$ is the energy needed to excite an H atom from its ground state to the $n = 2$ level. It is 10 eV.

**Exercise 9.3:**

Here are two numerical problems that also fold in the application of some simple physical thinking:

(i) Estimate the mass of the Sun given that the distance of the Earth from the Sun is $1.5 \times 10^{11}$m and the Earth takes one year to complete an orbit of the Sun. (Hint: as this is an estimate, you are fully justified in using a simplified physical model that you are already well used to.)

(ii) A typical maximum momentum $p_{max}$ for the free electrons in a metal where

$$p_{max} = (\frac{3n_{e,v}h^3}{8\pi})^{1/3}$$

(In estimating $n_{e,v}$, the number density of free electrons, assume there is one such electron associated with every atom in the metal, and either look up or guestimate how dense typical metals are compared with water – for which you should know the density.)

From here on, you can expect to have to start applying your core knowledge of physics. Just what that core might be will be discussed and pulled together in connection with hand-out C. Before we do that – some even rougher estimations.

## 10 ‘Order of magnitude’ calculations

A factor of 10 is considered to be an order of magnitude, so an ‘order of magnitude’ calculation is one in which the answer is calculated to the correct power of 10. If the answer is in the form a number times $10^n$, then the value of $n$ is correctly calculated. The concept of an order of magnitude calculation is imprecise, but it usually means that the answer is correct to within maybe a factor of 3 (=$\sqrt{10}$). But it may not be this good. When working at this level of approximation, be ”bloody, bold, and resolute” about having to guess values for some of the quantities involved – have faith that over-estimation of one component quantity and under-estimation of another can end up in compensation... The shame is in not trying, rather than in getting it ’wrong’ (whatever that means).

To build confidence for this style of estimation, it helps to start compiling your own list of ’useful numbers’ to commit to memory that enables you to set scales to a wide variety of physical phenomena. For example any physicist deserving the name should not have to think what the mass density of water is, or what would be a typical atomic radius, the wavelength of visible light, etc. In the following you will have to draw on everyday experience, beyond the narrowly academic.

**Exercise 10.1:** From your rough ideas or knowledge of the density and radius of the earth, estimate its mass to an order of magnitude.

**Exercise 10.2:** Estimate the mass of water in all the World’s oceans.

**Exercise 10.3:** Estimate the impulse due to a raindrop hitting the ground.

**Exercise 10.4:** Estimate the temperature at which the Earth would immediately lose most of its atmosphere.

**Exercise 10.5:** Estimate the number of midwives needed in the Greater London area? AND/OR.. How many piano tuners can be fully-employed in the Greater London area?

**Exercise 10.6:** A hiker is walking along the cliff path near Lyme Regis when he comes across a trackway of dinosaur footprints. As he examines them, he becomes thirsty and takes a mouthful out of his water bottle. How many water molecules in the drink he took are likely to have come from the dinosaur who made the footprint?

**Exercise 10.7:** How many kilograms of helium are produced every second in the Sun?

**Exercise 10.8:** A proton has a size $a$ of 1 fm, and the size of a nucleus scales as $a \times A^{1/3}$, where $A$ is the number of nucleons (protons and neutrons) making up the nucleus. Assuming that it takes a direct collision with a nucleus to stop a proton, derive an order-of-magnitude estimate for the range of a proton in lead.

# 11 Identifying whether a physical effect is important – the 'back of the envelope'

When you approach an unfamiliar problem, the first step is to identify the important physics. Suppose you were a physicist in the early 20th century before the discovery (or invention?) of Quantum Mechanics. You know of the electron and the proton and their masses. You want to develop a theory of the atom.

As a first step you could prove that the atom cannot be held together by gravity. To do this you need to formulate the question in a quantitative way, showing either that some quantity is very large or very small compared with another quantity. You could show that the gravitational force between an electron and a proton at some spatial separation (which you must estimate) is small, but it has to be small compared with another relevant force. Alternatively, you could show that the gravitational potential energy at the same separation is small compared to another relevant energy. The potential energy could be shown to be small compared with the thermal energy ($kT$), which is the minimum energy the electron could be expected to have.

The energy argument looks most promising – especially as it persuasively contrasts a negative binding energy with a positive kinetic energy. The characteristic size $R$ of an atom can be estimated from the mass $m_p$ of a proton and the density $\rho$ of a solid:$R \sim (m_p/\rho)^{1/3}$. The gravitational potential $\epsilon_G$ of an electron a distance $R$ from the proton is $\epsilon_G \sim Gm_pm_e/R$. Atoms are stable at room temperature, so gravity can only hold an atom together if $\epsilon_G$ is comparable with or larger than the thermal energy $kT$, which is therefore an estimate of the kinetic energy, at room temperature. Put the numbers in to show that this condition is not satisfied. You will not need a calculator to prove this.

You could now go on and show that electrostatic attraction can do the job in a way gravity cannot – provided, of course, that the physics of your time allows you to know the magnitude of the charge on a single electron or proton.

The exercises below are further comparisons of relative magnitude of effect. The calculations involved need be nothing more than crude 'order of magnitude' estimates given that that the aim is demonstrate or rule out large differences. Estimation of this type is what is often referred to as 'back of the envelope'. The idea behind this figure of speech is that, when you have that prize-winning stroke of scientific inspiration sitting on a train or in a cafe, you reach for the only piece of paper to hand (an envelope...?) to estimate whether your inspiration even *begins* to work on the right physical scale.

> **Exercise 11.1:** Demonstrate the truth, or untruth, of the following statements. These require you, first, to identify the relevant physics, before making numerical estimates.
>
> (i) Switching on car lights does not noticeably affect a car's performance.
>
> (ii) Energy released by industrial activity cannot be directly responsible for global warming.
>
> (iii) The sun cannot be powered by chemical reactions.