Professional Skills for Physicists: II

PROBLEM SOLVING

Introduction and section A

January – March 2008

Introduction

The importance of problem-solving

An ability to solve problems of all kinds is one of the most valuable skills learnt by physics students. This includes not only problems directly related to physics, but any problem without a clear starting point or way in. Physics students should excel at problems requiring a numerical approach, but even physics problems require a wider range of skills such as visualisation in three dimensions and learning to pick out the important factors whilst ignoring the less important. This series of seminars aims to

- Introduce you to some of the building blocks of problem solving in physics
- Acquaint you with the value of approximate and order-of-magnitude techniques
- Develop your ability to tackle unfamiliar, unstructured problems

Be warned, you will not necessarily be provided with all the information needed to solve all the problems included in this booklet – we want you to begin to pick up the strategies (e.g. creative guesswork, reality-checking, research) that will move you on from spoon-fed problem-solving.

The expectation is that you will work through parts of this booklet, and others supplied later on, under the supervision of and at a pace set by your seminar leader. The priority for you should be to use the seminars, working both on your own and co-operatively, to practise problem-solving and gain in understanding and confidence. At intervals, you will move on to a new handout and section of work. Exercises left undone and unread material are there to be attended to in your own time. On a best efforts basis, seminar leaders will respond to queries arising from private study brought back to seminars.

There will be no distribution of photocopied 'model answers'. From time to time, solutions will be presented on the whiteboard at the discretion of your seminar leader, and bottom-line answers to individual questions will usually be available. You, or a representative of the small group you are working in at the time, may be asked to present solutions at the seminar room board. Please be aware that many problems can be tackled in more than one way.

This series of seminars is intended to promote generic skills that will help you become the kind of flexible problem-solver that does well in the two 3-hour Physics comprehensive papers. These are sat in year 3 by most students – those who spend a year in Europe take them in year 4.

A rough break down of the 8.5 seminars up to the end of term is:

- Dimensional analysis – 3 weeks (hand-out A – this one)
- Numerical estimation and approximate methods – 3 weeks (hand-out B)
- Useful tools (especially energy), drawing diagrams and idealised models – 3 weeks (hand-out C)
- Longer problems, including a past test paper (hand-out D)

Problem-solving test

The time and place for this will be:

Friday, May 2nd 10.00 – 11.30
Blackett lecture theatres (as for preceding exams)

The test paper will be made up of a compulsory section based on core mastery-level material, followed by a section giving you the choice of one out of three questions. No calculators are to be used. You will be supplied with a list of the common physical constants and other frequently needed quantities. 50 percent of the marks go to each section.

Well laid out, properly-reasoned solutions will earn you full marks: just getting the right answers will earn marks, but certainly not full marks. In marking the test answer books, we will want to be able to follow your working out, so we can be sure you know what you are doing.

A) Dimensional Analysis

1 Dimensions

When you find a numerical answer to a problem, you will be aware that you should always give the units. For example, if the answer is a velocity, you give the answer in terms of metres per second. You might also give the answer in terms of cm s^{-1}, feet s^{-1} or miles hour^{-1}. Whatever system of units you choose, the unit for velocity is always a length divided by a time. Accordingly, the dimensions of velocity are length divided by time. One way of writing this is: [velocity]=LT^{-1} where L stands for length and T for time. Similarly, [acceleration]=LT^{-2} and, since energy can be written as a mass times a velocity squared, [energy]=ML^{2}T^{-2}, where M stands for mass.

A general method for finding the dimensions of a quantity is to write down an equation that includes it, in which all other quantities present have known dimensions. For example, the dimensions of energy (ϵ) can be found from the equation $\epsilon = \frac{1}{2}mv^2$ or equivalently from the equation $\epsilon = mc^2$.

> **Exercise 1.1:** Use the following equations to find the dimensions of force (*F*), momentum (*p*), number density (*n*), mass density (ρ), pressure (*P*) and energy density (*U*):
>
> $$F = ma \quad p = mv \quad n = \frac{N}{(4/3)\pi R^3}$$
>
> $$\rho = nm \quad P = F/A \quad U = \frac{1}{2}nmv^2$$
>
> where a is acceleration, N is a number (dimensionless), and A is an area.

Some quantities are dimensionless – for example N above, which happens to be the total number of particles in the specified volume.

A further common dimensionless quantity is angle. Although we give angles units of degrees or radians, these are in fact dimensionless. This is clear from the way we express an angle in radians as the length of an arc over a radius, ie an angle is a length divided by a length, and therefore dimensionless.

Exercise 1.2: Use the following equations to find the dimensions of angular velocity (ω), angular momentum (L), a couple (τ), and moment of inertia (I):

$$\omega = \theta/t \quad L = mrv \quad \tau = Fr \quad L = I\omega \;\; (\text{or } I = L/\omega)$$

where θ is angle, t is time, r is radius, v is velocity and F is a force.

Exercise 1.3: (i) Use the equation $\epsilon = h\nu$, where ν is the frequency of radiation, to show that Planck's constant h has dimensions of angular momentum. (ii) Find the dimensions of the gravitational constant G.

2 Dimensions in differential and integral equations

A differential dY/dX is the limit of the division of a small change in Y (ΔY) by a small change (ΔX) in X. dY/dX therefore has the same dimensions as Y/X.

Exercise 2.1: Using the results and definitions in section 1, find the dimensions of dv/dt, $d\epsilon/dx$ and $\sqrt{dP/d\rho}$, and show that they have the same dimensions as acceleration, force and velocity respectively.

A similar principle applies to integrals. An integral $\int f(x)dx$ can be understood as the area under a curve. $f(x)$ gives the vertical height and dx is the increment in the horizontal dimension. Hence the dimension of the integral is the dimension of f times the dimension of x. For example, the work W done by a force F pushing through a distance x is $W = \int F dx$ and $[W] = [F][x]$.

Exercise 2.2: Using the results and definitions in section 1, find the dimensions of $\int \rho dV$, $\int \tau d\theta$ and $\int U dV$, where $\int dV$ represents an integration over a volume. You should find that the dimensions of the integrals are mass, energy and energy respectively.

Since a second-order differential such as $d^2Y/dX^2 = d(dY/dX)/dX$, its dimensions are $[Y][X]^{-2}$. A familiar case is acceleration that is a second differential of length wrt time and so has dimension $[L][T]^{-2}$.

3 Dimensions and temperature

This is a first example in which the concept of energy helps us to bring into a dimensional system some of the more abstract physical quantities that have no immediately obvious expression in terms of mass, length or time.

Temperature is a measure of average energy. The way in which you probably have seen this connection made is through the product $\frac{1}{2}fkT$ which is the mean energy of a gas molecule in an ensemble of molecules at temperature T. The multiplying factor k is the Boltzmann constant, an example of what is known as a *physical* constant – as opposed to a dimensionless *numerical* constant like f, here representing the number of degrees of freedom each molecule has. Since kT is required to have the dimensions of energy, k must be an energy divided by a temperature. More explicitly, since $[kT] = \mathrm{ML}^2\mathrm{T}^{-2}$), and the dimension of T is, shall we say, K for Kelvin, its SI unit of measurement ($[T] = \mathrm{K}$), the dimensions of Boltzmann's constant must be $[k] = \mathrm{ML}^2\mathrm{T}^{-2}\mathrm{K}^{-1}$.

> **Exercise 3.1:** Show that nkT has the same dimensions as both pressure and energy density. Here, n represents the particle number density.

4 Dimensions and electrostatics

In electrostatics we have to introduce two new quantities:
(i) charge with dimension Q and units of Coulombs (SI), and
(ii) electric potential measured in units of Volts (SI).
Again, we can make sense of them dimensionally through the concept of electrical energy: we know that a charge q in an electric potential V has an energy qV. Hence $[\mathrm{charge}] \times [\mathrm{potential}] = \mathrm{ML}^2\mathrm{T}^{-2}$, or, if we want to separate out the dimension of electric potential, we can write $[V] = \mathrm{ML}^2\mathrm{T}^{-2}Q^{-1}$.

> **Exercise 4.1:** The equation of motion for an electron, carrying charge e, in an electric field E is $mdv/dt = eE$. From this equation show that $[E] = \mathrm{MLT}^{-2}Q^{-1}$. Alternatively, from the equation $E = -dV/dx$ show that $[E] = [V]L^{-1}$. Confirm that these two expressions for $[E]$ imply the same expression for $[V]$ as given in the sentence immediately preceding this exercise.

Exercise 4.2: In SI units, the energy density of an electric field is $\frac{1}{2}\epsilon_0 E^2$. The dimensions of energy density (U) were calculated in exercise 1.1, and the dimensions of E were calculated in exercise 4.1: Hence show that dimensions of ϵ_0 are $[\epsilon_0] = \mathrm{M}^{-1}\mathrm{L}^{-3}\mathrm{T}^2 Q^2$.

Exercise 4.3: In SI units, the force between two electrons separated by a distance r is $F = (1/4\pi\epsilon_0)(e^2/r^2)$. Confirm that this gives the same dimensions for ϵ_0 as obtained in exercise 4.2.

5 Some useful tricks

Temperature (usually measured in Kelvin) always occurs in equations either in conjunction with Boltzmann's constant k or Stefan-Boltzmann's constant, σ. Examples of this are the equation for pressure, $P = nkT$, the equation for energy density $U = \frac{3}{2}nkT$ and the equation for energy flux $F = \sigma T^4$. T mostly appears as kT to make an energy, and can only appear apart from k or σ when it appears as a ratio against another temperature.

The same applies to electrostatic quantities. Charge (q), potential (V) and ϵ_0 can only appear in certain combinations. qV is an allowed combination because it makes an energy. q^2/ϵ_0 is allowed because it occurs in the electrostatic force equation $F \propto e^2/(r^2\epsilon_0)$. $\epsilon_0 E^2$ is an energy density, while the electric field E is a potential V divided by a distance. Consequently $\epsilon_0 V^2$ is an allowed combination.

These special combinations can be used to simplify a dimensional analysis. For example, a well known distance scale in plasma physics is the Debye length, $\lambda_D = (\epsilon_0 kT/ne^2)^{\frac{1}{2}}$. The easy way to show that this expression really does have the units of length is to group terms:

$$\lambda_D = \left(\frac{\epsilon_0 kT}{ne^2}\right)^{\frac{1}{2}} = \left(kT \times \left(\frac{r^2\epsilon_0}{e^2}\right) \times \left(\frac{1}{r^2}\right) \times \left(\frac{1}{n}\right)\right)^{\frac{1}{2}}$$

where we have introduced an extra r^2 in two places. The reason for doing this is that $e^2/(r^2\epsilon_0)$ is a force (mathematically this is allowed because the two appearances of r^2 cancel, and we do not even have to say what distance r actually is). We can now use the simplifications

$$[kT] = [\text{energy}] = [\text{force}] \times [\text{distance}] \qquad [\frac{r^2\epsilon_0}{e^2}] = [\text{force}]^{-1}$$

$$[\frac{1}{r^2}] = [\text{distance}]^{-2} \qquad [\frac{1}{n}] = [\text{distance}]^3$$

to show that λ_D does indeed have the dimensions of distance.

Exercise 5.1:

$$N_D = \frac{4\pi(\epsilon_0 kT)^{\frac{3}{2}}}{3n^{\frac{1}{2}}e^3}$$

Show that N_D is dimensionless.

Exercise 5.2: According to Bohr's theory, the ionisation energy of hydrogen is

$$\epsilon_i = \frac{m_e e^4}{8\epsilon_0^2 h^2}$$

Show that ϵ_i has dimensions of energy.

6 Dimensions and magnetic field

Magnetic field is in many ways analogous to electric field and must always appear in certain combinations. As with electric field, dimensional analysis is best conducted by grouping magnetic field in combinations with recognisable dimensions. Useful equations for dimensional analysis including magnetic field and related quantities are, in SI,

- The force F on an electron moving at velocity v across a magnetic field B is $F = evB$.
- The energy density of a magnetic field is $U = B^2/2\mu_0$
- An electron, with mass m_e, executes a circular motion in a magnetic field with a period $t_B = 2\pi m_e/eB$.
- $[E]/[B] = [\text{velocity}]$ (see exercise 6.2 below)

If these combinations are used, there is no need to worry about the actual dimensions of B, although the first and last of the above equations show that $[B] = \mathrm{MT}^{-1}\mathrm{Q}^{-1}$.

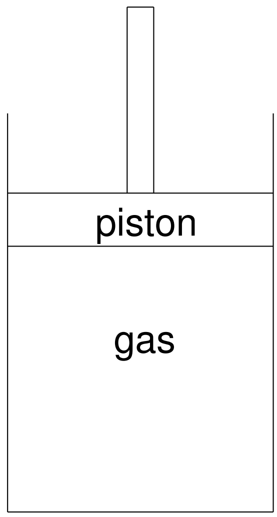
Exercise 6.1: Show that $v_a = B/\sqrt{\rho\mu_0}$ has dimensions of velocity.

Exercise 6.2: For an electromagnetic wave propagating in the x direction, the electric field (in the y direction) and magnetic field (in the z direction) are related by the partial equation $\partial E/\partial x = -\partial B/\partial t$. Show that E/B has dimensions of velocity.

Exercise 6.3: Show that $1/\sqrt{\mu_0\epsilon_0}$ has dimensions of velocity.

7 Using dimensions to check equations

Mistakes are easily made when manipulating mathematical equations. Dimensional analysis provides an easy way of checking for errors. To concoct a rather contrived case, imagine a gas confined within a vertical tube by a freely-moving piston.



The total energy ϵ_{total} in the system can be written as

$$\epsilon_{total} = \epsilon_{ke} + \epsilon_{grav} + \epsilon_{thermal} + \epsilon_{motion}$$

where ϵ_{ke} is the kinetic energy of the moving piston, ϵ_{grav} is the gravitational energy of the piston which changes as the piston moves vertically, $\epsilon_{thermal}$ is the thermal energy of the gas in the cylinder which changes as the piston compresses it, and ϵ_{motion} is the bulk kinetic energy of the moving gas. One might actually find other contributions to the total energy of the system such as frictional heating. The energy equation only makes sense if every term in the equation has dimensions of energy. This can be checked by a dimensional analysis of each term. If an error is made in deriving the equation, this may well show itself as an error in the dimensions in the incorrect term. This applies not only to energy equations, but to all equations in physics, and this leads to the following rule:

RULE: All terms added together in an equation must have the same dimensions

Although dimensional analysis will not reveal errors due to missing numerical factors such as $4\pi/3$, it will reveal many errors in algebraic manipulation. Hence, if you are engaged in some difficult maths, it pays to perform an occasional dimensional analysis to check you are staying on track.

Another useful check from time to time, is to remember that arguments of certain functions are of necessity dimensionless. An obvious example of this is provided by the trigonometric functions (sine, cosine...) that demand angles as arguments – it was noted in section 2 that angles are without dimension since they are formed from the ratio of two lengths. Important further instances of functions requiring dimensionless arguments are the exponential and logarithmic functions: many physical processes involve exponential decay or growth which leads to their frequent appearance. Since the arguments of exponential or log functions can themselves be quite complicated expressions, it can help to check that they are dimensionless as required: if they are not, you will know you have gone wrong somewhere and have to reconsider.

Exercise 7.1: Ascertain whether the following equations are dimensionally plausible:

(i) $v^2 = GM/r^2$

(ii) $kT = eE$

(iii) $\epsilon_0 E = B/\mu_0$

(iv) $e^2/4\pi\epsilon_0 = GM^2$

Exercise 7.2: Examine the following equations for dimensional plausibility:

(i) The time-dependent Schrodinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + U(x)\psi = i\hbar\frac{\partial\psi}{\partial t}$$

where $\psi(x,t)$ is the wavefunction, which varies in space and time, and $U(x)$ is the potential energy of a particle at position x.

(ii) The Maxwell equation for electric field E in the y direction with a magnetic field B in the z direction

$$-\frac{\partial B}{\partial x} = \mu_0 j + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

where j is the current density (current divided by cross-sectional area) in the y direction. You can use results derived in section 6.

8 Using dimensions to derive equations

A sphere of radius R moving through a gas with density ρ at velocity v is subject to a decelerating force F. Dimensional analysis of these four quantities gives $[R]$=L, [ρ]=ML^{-3}, $[v]$=LT^{-1} and $[F]$=MLT^{-2}. Common sense tells us that F depends on some combination of R, ρ & v. The only possible dimensionally correct dependence of F on these quantities is $F = \text{constant} \times \rho^\alpha v^\beta R^\gamma$ where α, β & γ are chosen to make the equation dimensionally correct. The left hand side of the equation has dimensions $[F]$=MLT^{-2}, And the right hand side has dimensions $[\rho]^\alpha [v]^\beta [R]^\gamma = (ML^{-3})^\alpha (LT^{-1})^\beta R^\gamma = M^\alpha L^{-3\alpha+\beta+\gamma} T^{-\beta}$.

For dimensional validity, the left and side and the right hand side must have the same dimensions, giving

$$\mathrm{MLT}^{-2} = \mathrm{M}^\alpha \mathrm{L}^{-3\alpha+\beta+\gamma} \mathrm{T}^{-\beta}$$

Considering the powers of M, L and T separately, we can extract three simple simultaneous equations:

$$\alpha = 1$$
$$-3\alpha + \beta + \gamma = 1$$
$$\beta = 2$$

From these, we have $\alpha = 1$, $\beta = 2$ and $\gamma = 2$, giving:

$$F = \text{constant} \times \rho v^2 R^2$$

Dimensional analysis cannot tell us the value of the numerical constant, but there is no other way of combining these quantities in a dimensionally correct expression, So this must be the equation for the force on the sphere. The equation tells us that if the gas density is doubled the force is doubled, and if either the velocity or the radius is doubled the force is increased by a factor of four.

Exercise 8.1: A supernova explosion launches a shock wave moving at velocity v into the interstellar gas with density ρ. The gas pressure immediately behind the shock is P. Show by dimensional analysis that P=constant $\times \rho v^2$.

Exercise 8.2: A pendulum consists of a mass m swinging on the end of a massless string of length l. Given that the frequency ω of a small-amplitude oscillation is independent of its amplitude, show that

$$\omega = \text{constant} \times \sqrt{\frac{g}{l}}$$

How does the period of oscillation change if the length of the string is doubled?

Exercise 8.3: Given that the velocity of sound in a gas (consisting of only one kind of molecule) may depend only on the temperature, the mass of the molecules, and the number density of molecules, work out what the dependence of the sound velocity on these quantities should be. Note: recall the earlier discussion of how temperature in Kelvin can occur only in combination with Boltzmann's constant.

Exercise 8.4: Use the result of exercise 8.3 to explain why inhaling helium raises the pitch of the voice.

Dimensional analysis only works for exercise 8.2 if the string is massless, otherwise there are two masses, that of the mass m and that of the string, and there are other ways of combining quantities into a dimensionally correct equation. Similarly in exercise 8.3, dimensional analysis fails if the gas is constituted of molecules with a range of masses.

To finish, a longer, more challenging piece of analysis.

Exercise 8.5: The flux of radiation (energy passing through unit area in unit time) from a blackbody (a source in thermal equilibrium) at temperature T is σT^4, where σ is the Stefan-Boltzmann constant: its value is 5.7×10^{-8} W m^{-2} K^{-4}. It is also possible to write σ in terms of the better-known physical constants k, h and c as:-

$$\sigma = \mu k^{\alpha} h^{\beta} c^{\gamma}$$

where μ, α, β and γ are numerical constants. What are the values of α, β and γ? Show by rough calculation that μ is $\sim$40.