Pigeonhole I16

Mathematical Methods II — Dr R. E. Hunt
Natural Sciences Tripos Part IB — Lent 2002

Hints and Solutions for Example Sheet 4: Complex Analysis, Contour Integration and Transform Theory

1 The imaginary parts are $\cos x \sinh y$, $-e^{y^2-x^2}\sin 2xy$ and $-y/(x^2+y^2)$ respectively (ignoring arbitrary constants). Hence the complex functions are $\sin z$, $\exp(-z^2)$ and $1/z$.

2 Define $\Phi(x,y) = -E\,\mathrm{Re}(z - a^2/z)$ where $z = x+iy$. Note that $\nabla^2\Phi = 0$. Write $z = re^{i\theta}$, and hence show that $\Phi = 0$ on $r = a$. Find also an approximate expression for Φ as $r \to \infty$ and deduce that at large distances, $-\nabla\Phi$ has magnitude E in the x-direction.

3 $\log\tanh z$ is analytic except when $\tanh z$ is real and negative: show that this does not occur in the domain. For the bar of width L, the temperature is

$$\mathrm{Im}\left\{\frac{2T_0}{\pi}\log\tanh\frac{\pi z}{2L}\right\}.$$

4 $(z-i)^2/(z+1)$ has a double zero at $z = i$ and a simple pole at $z = -1$.

$(1+z)^{-1} - (1-z)^{-1}$ has a simple zero at $z = 0$ and simple poles at $z = \pm 1$.

$(z^2+i)^{-1}$ has no zeros but simple poles at $z = \pm e^{-\pi i/4}$.

$\sec^2 \pi z$ also has no zeros but double poles at $z = n + \frac{1}{2}$ (where n is any integer).

$\sin z^{-2}$ has simple zeros at $z = \pm 1/\sqrt{n\pi}$ for positive integers n, and a (non-isolated) essential singularity at $z = 0$.

$\sinh\{z/(z^2-1)\}$ has simple zeros at $z = 0$ and at

$$z = -\frac{i}{2n\pi} \pm \sqrt{1 - \frac{1}{4n^2\pi^2}}$$

(from solving $z/(z^2-1) = n\pi i$) for any non-zero integer n. It has essential singularities at $z = \pm 1$ (its growth is exponentially fast near each of the singularities).

$(\tanh z)/z$ has zeros at $z = m\pi i$ for non-zero integers m, simple poles at $z = (n+\frac{1}{2})\pi i$ for integers n, and a removable singularity at $z = 0$.

5 Part (i) was proved in lectures, and parts (ii)–(iv) follow from it. For part (v), write down the Laurent expansion of f and substitute it into the given formula.

6 $(z+1)/z^2$ has a double pole at $z=0$ with residue 1; e^{-z}/z^3 has a pole of order 3 at $z=0$ with residue $\frac{1}{2}$; and $\sin^2 z/z^5$ has a pole of order 3 at $z=0$ with residue $-\frac{1}{3}$ (use the Taylor expansion of sin). There are simple poles of $\cot z$ at $z=n\pi$ (n an integer), with residue 1 at each pole (use part (iv) of the previous question). Finally, $z^2/(1+z^2)^2$ has double poles at $z=\pm i$, with residues $\mp\frac{1}{4}i$ (use part (v) of the previous question).

7 There are infinitely many possibilities; some are shown below. The values of the function on either side of the cuts are marked ($x=\operatorname{Re} z$, $y=\operatorname{Im} z$).

8 Most of this question has been done in lectures. For the final equation, apply Cauchy's formula to the function $f'(z)$.

9 Use Cauchy's formula differentiated n times with respect to z_0; and find a bound for the integral round the circle. To prove Liouville's Theorem, set $n = 1$ and observe that the formula is true for all r; and also for all z_0.

10 Proceed as for the trigonometric functions worked example in lectures. There is a pole inside the contour at $z = a^{-1}$ with residue $(a^{n-1}-a^{n+1})^{-1}$. The answer is $2\pi/a^n(a^2-1)$.

11 Proceed as for the Fourier transform worked example in lectures; take the real part to obtain the given equality. The general result, valid for both positive and negative k, is $\pi e^{-|k|}$, because it must be an even function of k.

12 (i) Use a keyhole contour, as in the branch cut worked example in lectures.

(ii) Proceed as for the trigonometric functions worked example in lectures. Note that the upper limit of the integral is only π, not 2π, but that you can double it up. You should obtain a fourth-degree polynomial on the denominator; this must have roots when $\sin\theta = \pm ia$ (why?). Find these roots and show that two are inside the unit circle. [*A better (quicker) method is to first replace* $\sin^2\theta$ *by* $\frac{1}{2}(1-\cos\phi)$ *where* $\phi = 2\theta$*; then the polynomial on the denominator is only quadratic.*]

(iii) If you use a semicircular contour, then there are four poles within the contour. At any of these poles (say z_0), the residue is $1/8z_0^3$ (using L'Hôpital's Rule). The sum of the residues is $\frac{1}{4}i(\sin\frac{\pi}{8} - \sin\frac{3\pi}{8})$, and (believe it or not) each sin can be expressed in terms of the square root which appears in the answer. If instead you use the suggested approach of a contour which forms a sector of a circle, only one pole is enclosed and the answer is considerably easier to obtain.

(iv) Integrate $\exp(\frac{1}{2}iaz^2)$ round a sector of a circle of angle $\pi/4$. One of the three resulting integrals can be expressed as a standard real integral, for which you know the answer. It is difficult to prove that the integral round the large circular arc vanishes in the limit; to do it formally you would need to use the method used to prove Jordan's Lemma.

(v) Proceed as for the worked example in lectures with a singular point on the axis. You will need to define a branch cut for $\log z$, which can be taken either along the positive real axis or along the negative imaginary axis (say); in either case the value of $\log z$ on the positive real axis is simply $\log x$. The contribution from the negative real axis has three parts; one is the same as the contribution from the positive real axis, one is purely imaginary (and so can be removed by taking real parts) and one is a standard integral.

13 Proceed as for the Fourier transform worked example in lectures. The required function is

$$\tfrac{1}{2}e^{-|x|/\sqrt{2}}\sin\left(\frac{\pi}{4}+\frac{|x|}{\sqrt{2}}\right).$$

14 Integrate $\oint \operatorname{sech} z \, dz$ around the given countour. The real axis gives you $2I$ where I is the required integral. When $z = i\pi + x$, show that $\operatorname{sech} z = -\operatorname{sech} x$, so that the upper side of the rectangular contour gives $2I$ as well. The two vertical sides of the rectangle give zero as $R \to \infty$, which you can demonstrate by showing that $|\cosh(R+iy)| \geqslant \sinh R$.

The only singularity of the integrand within the contour is at $z = i\pi/2$, where it has a simple pole with residue $-i$. The result follows.

15 (i) $(p-\alpha)^{-1}$; $n!/p^{n+1}$; $p/(p^2-\alpha^2)$; $2\alpha p/(p^2+\alpha^2)^2$; $e^{2(1-p)}/(p-1)$; $(e^{-ap}-e^{-bp})/p$.

(ii) For the penultimate inversion, take care over in which direction you close the inversion contour. For the final inversion, split it into two parts and invert each separately.

16 (i) Use partial fractions before inverting. The answer is $x = 3t + 2\sin 2t$.

(ii) You should obtain a first order differential equation for $\bar{x}(p)$, which you can solve using an integrating factor to obtain $\bar{x}(p) = p^{-2} + ce^{p^2/2}$, where c is an arbitrary constant. Evaluate c by considering the limit of $p\bar{x}(p)$ as $p \to \infty$. The answer is $x = t$.

17 Find $\bar{f}(p)$ (you should obtain $(p+1)^{-1}e^{-p-1}$). Take the Laplace transform of each equation and solve the resulting linear simultaneous equations to obtain

$$\bar{x} = \frac{p+5}{(p+1)(p+2)(p+4)}e^{-p-1},$$
$$\bar{y} = \frac{3}{(p+1)(p+2)(p+4)}e^{-p-1}.$$

Use partial fractions before inverting to obtain the solution

$$x = \tfrac{4}{3}e^{-t} - \tfrac{3}{2}e^{1-2t} + \tfrac{1}{6}e^{3-4t},$$
$$y = e^{-t} - \tfrac{3}{2}e^{1-2t} + \tfrac{1}{2}e^{3-4t}$$

for $t > 1$ (and $x = y = 0$ for $t < 1$).

18 To obtain the expression for $x(t)$, find $g(t)$ by inverting $\bar{g}(p)$ and then write down the convolution $f * g$. For the final part, substitute $f(\tau) = \delta(\tau)$ into the convolution.

Comments on or corrections to this problem sheet are very welcome and may be sent to me at `reh10@damtp.cam.ac.uk`.