Pigeonhole I14

Mathematical Methods II — Dr R. E. Hunt
Natural Sciences Tripos Part IB — Lent 2002

Hints and Solutions for Example Sheet 3: Cartesian Tensors

1 (i) 3; $\mathbf{a}$; $|\mathbf{a}|^2$; 3; 0; 0; $\mathbf{b}\,.\,(\mathbf{c}\times\mathbf{a})$; $-\det A$.

(ii) $\mathbf{x} = (\mathbf{b}\,.\,\mathbf{c})\mathbf{a} + \mathbf{d}$; invalid; $u = \mathbf{x}\,.\,(\mathbf{v}\times\mathbf{w})$; $|\mathbf{x}\times\mathbf{y}|^2 = 1$; $AB = TI$; $\mathbf{x} = AB^T\mathbf{y}$.

(iii) There are many possible solutions, differing in which suffixes are used. Sample answers are

$$\begin{aligned}(x_i + \mu y_i)(x_i - \mu y_i) &= 0;\\ x_i &= a_j a_j b_i - b_j b_j a_i;\\ 2\epsilon_{ijk} x_j y_k (a_i + b_i) &= \lambda;\\ a_{ij} x_j &= b_i - b_{ji} y_j;\\ x_i y_i a_{jj} &= 3|x_i y_i|;\\ a_{ij} x_j a_{ik} x_k &= \frac{1}{x_i y_i} y_j a_{kj} b_{kl} c_{lm} x_m.\end{aligned}$$

2 (i) Use the fact that the suffixes of ϵ_{ijk} can be rearranged cyclically.

(ii) Start by proving the equation given in lectures relating $\epsilon_{ijk}\epsilon_{lmn}$ to a determinant, following the method outlined there. $\epsilon_{ijk}\epsilon_{ijk} = 6$.

3 Transform the inertia tensor using the matrix formulation of the transformation law. In the second frame, it has components

$$\tfrac{1}{2}\begin{pmatrix} 5 & 0 & \sqrt{3} \\ 0 & 4 & 0 \\ \sqrt{3} & 0 & 3 \end{pmatrix}.$$

To find the components of I in the third frame, you will first need to find the components of $\mathbf{e}_3$ relative to the new axes (using orthogonality). Then find the rotation matrix using what you know about its columns. The resulting transformed components of I are

$$\tfrac{1}{3}\begin{pmatrix} 5 & -2 & -1 \\ -2 & 5 & 1 \\ -1 & 1 & 8 \end{pmatrix}.$$

$I_{ij}I_{ji} = 14$; note that this is a scalar and can be evaluated in any frame!

4 Evaluate each of the components of I_{ij} separately. Write down their definitions explicitly in terms of x, y and z; symmetries of the cylinder mean that you only need to calculate four of the components. $I_{ij} = \frac{1}{2}Ma^2\delta_{ij}$; it is coincidence that I is isotropic.

5 Extend the method given for second rank tensors in lectures (during the initial discussion of the conductivity tensor).

6 (i) Consider particular values of i, j and k.

(ii) Remember that ϵ_{ijk} is *defined* independently of the frame, just as δ_{ij} is. Try the transformation law on ϵ_{ijk} and use part (i).

(iii) Either calculate $\det A'$ in suffix notation and use part (i) twice; or (quicker) use a matrix relationship between A' and A.

7 Split the tensor S_{ij} from lectures into a part with zero trace and a multiple of the identity matrix (which is isotropic, being δ_{ij}).

The given matrix decomposes as

$$\begin{pmatrix} 3 & -1 & 5 \\ 1 & 0 & 5 \\ 1 & -5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 5 \\ -2 & -5 & 0 \end{pmatrix}.$$

The symmetric zero-trace part has eigenvalues -2, -2 and 4, which sum to zero: remember that for any matrix A, $\operatorname{Tr} A$ is a scalar.

8 The $\boldsymbol{\omega}$ of the question is the dual vector of A as described in lectures.

Observe that $B = A^2$. Calculate $B\mathbf{x}$ and find the eigenvalues of B by inspection: they are 0, $-|\boldsymbol{\omega}|^2$ and $-|\boldsymbol{\omega}|^2$.

9 Consider the principal values. No current flows in the direction $(\sqrt{2}, -1, 1)$.

Evaluate $E_i J_i$ (which is a scalar) in a suitable frame (in which it has a particularly simple form). Either by inspection, or more formally using a Lagrange multiplier, you can show that the minimum dissipation rate is 0 and the maximum is 4 (for unit $|\mathbf{E}|$).

10 α, β and γ are scalars, so are isotropic, as is δ_{ij}, etc. (by definition).

To show that σ' is trace-free, contract i and j and use the definition of p. For an isotropic Newtonian fluid, there must be a fourth rank isotropic tensor relating the second rank tensors σ' and e. Note that e is symmetric. Remember to use the definition of p; the relationship between μ and α, β, γ turns out to be $\mu = -\frac{3}{2}\alpha = \frac{1}{2}(\beta + \gamma)$.

To show that $\sigma'_{ij} e_{ij}$ is non-negative, first write it in terms of the eigenvalues of e_{ij}, and then find its minimum possible value with respect to those eigenvalues (by partial differentiation).

11 The conductivity tensor is

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix}$$

if we choose axes where $\mathbf{e}_1$ is parallel to the axis of symmetry. You can write down the components of the current density vector $\mathbf{J}$, given that a current I is flowing along the wire. This enables you to find the electric field $\mathbf{E}$ in the wire, using the conductivity tensor. Now find the potential difference across the ends of the wire due to this electric field. Deduce the resistance using Ohm's law.

12 0; $4\pi\delta_{ij}$; $4\pi/3$ (multiply out and note any isotropic integrals).

13 You will need to use several times the fact that the contraction of a symmetric with an anti-symmetric tensor is zero.

14 Evaluate both F_i (where $\mathbf{F}$ is the force) and $\partial S_{ij}/\partial x_j$ where S_{ij} is as given.

Comments on or corrections to this problem sheet are very welcome and may be sent to me at `reh10@damtp.cam.ac.uk`.