Pigeonhole I10

Mathematical Methods II Dr R. E. Hunt
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Hints and Solutions for Example Sheet 1: Variational Methods

1 If the box has dimensions $w \times h \times d$ then you need to minimise $w(h+d) + 2hd$ subject to $whd = V$. Using $\nabla\big(w(h+d) + 2hd - \lambda whd\big) = \mathbf{0}$ you should obtain $w = 2h = 2d$; the constraint gives $w = \sqrt[3]{4V}$.

2 You must find the stationary values of T both within the sphere and on its surface. Within the sphere, simply use $\nabla T = \mathbf{0}$ to show that *any* point with $x = 0$, $y = -z$ is stationary; at such a point, $T = 0$. On the surface of the sphere, use the constraint $x^2 + y^2 + z^2 = 1$ together with $\nabla\big(T - \lambda(x^2 + y^2 + z^2)\big) = \mathbf{0}$ to find four stationary points, $(\pm\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2})$ and $(\pm\frac{1}{\sqrt{2}}, -\frac{1}{2}, -\frac{1}{2})$, at which T takes the values $\pm\frac{1}{\sqrt{2}}$. Considering all of the stationary points (both internal and on the surface) together, it is clear that the minimum temperature is $-\frac{1}{\sqrt{2}}$ and the maximum $\frac{1}{\sqrt{2}}$.

(In fact, we will see later in the course that since $\nabla^2 T = 0$, it is not necessary to consider interior points.)

3 The length of a path is given by $\int \sqrt{1 + a^2\theta'^2}\,\mathrm{d}z$ if you use a path $\theta(z)$; or, equivalently, $\int \sqrt{a^2 + z'^2}\,\mathrm{d}\theta$ if you use a path $z(\theta)$. In either case, the analysis is similar to that for geodesics in the Euclidean plane, and you should find that z and θ are linearly related (e.g., $z = c + k\theta$ where c and k are constants).

4 The problem is essentially the same as the brachistochrone. Having shown that the optimal path is a cycloid, you need to find the corresponding value of the integral which you minimised in the first place; since the cycloid is parameterised by θ (say) you will need to substitute for x in terms of θ in the integral. To show that the tunnels are vertical at the end-points, consider $\mathrm{d}y/\mathrm{d}x = (\mathrm{d}y/\mathrm{d}\theta)/(\mathrm{d}x/\mathrm{d}\theta)$.

5 You need to minimise $\int \sqrt{a - bz}\,\sqrt{1 + z'^2}\,\mathrm{d}x$: use the first integral. After an elementary integration you should obtain

$$z = \frac{a}{b} - \frac{k^2}{4} - \frac{1}{k^2}(x - x_0)^2,$$

where k and x_0 are arbitrary constants, which is indeed an inverted parabola because it is of the form $(x - x_0)^2 = 4\alpha(z_0 - z)$ for suitable α and z_0. Using the hint, its directrix is $z_0 - z = \alpha$, which gives the required result.

6 The Lagrangian is $\mathscr{L} = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$ and the action is $\mathscr{S} = \int \mathscr{L}\,\mathrm{d}t$. Using the Euler–Lagrange equations we obtain

$$\ddot{r} - r\dot{\theta}^2 = -V'(r)$$

and

$$r^2\dot{\theta} = h,$$

a constant. These can be interpreted in terms of radial acceleration and angular momentum respectively.

If $r = a\sin\theta$ then conservation of energy gives that $\frac{1}{2}a^2\dot{\theta}^2 + V(r) = E$. Hence, using the second Euler–Lagrange equation, $V = E - a^2h^2/2r^4$; the result follows.

7 Consider $\int_A^B \mathrm{d}f$. Note that $\mathrm{d}f = \nabla f.\mathrm{d}\mathbf{r}$, and use an inequality for $\mathbf{a}.\mathbf{b}$ for any two vectors $\mathbf{a}$ and $\mathbf{b}$. To obtain the required condition for equality, recall that ∇f lies orthogonal to the family of surfaces $f =$ constant.

8 You must minimise $\int_{-b}^{b} 2\pi r\sqrt{1+r'^2}\,\mathrm{d}z$, where the soap surface is given by $r = r(z)$. The problem is effectively the same as the catenary. To show that the boundary condition has no solution for c if b/a is too large, try making the substitution $C = a/c$ and plotting appropriate graphs. As b/a is increased from below the critical ratio to above it, the soap film bursts and no longer joins the two circular wires.

9 You are required to maximise $\int_0^a y\,\mathrm{d}x$ subject to $\int_0^a \sqrt{1+y'^2}\,\mathrm{d}x = l$. Using a Lagrange multiplier and the first integral leads to

$$y' = \frac{\sqrt{\lambda^2 - (y-c)^2}}{y-c}$$

where c is a constant, which is easily integrated to give the equation of a circle. Thus the required curves are arcs of circles: note that it is not possible to know until you have finished the problem that your implicit assumptions about the shape of the curve will be satisfied so long as $l < \pi a$ but will fail otherwise!

10 Follow the derivation of Euler's equation in notes, replacing x and y by t and x respectively, and using a function $x(t) + \delta x(t)$. You will need to perform *two* integrations by parts for one of the terms.

For the last part, Euler's equation is $\frac{\mathrm{d}^2}{\mathrm{d}t^2}(t^4\ddot{x}) = 0$. Solving this differential equation with the given boundary conditions leads to $x(t) = t^{-2}$.

11 Remember to put the differential equation into self-adjoint form (trivial in this case). The Rayleigh quotient is

$$\Lambda = \frac{\int_{-1}^{1}(1+x^2)y'^2\,\mathrm{d}x}{\int_{-1}^{1}y^2\,\mathrm{d}x}.$$

The trial function y_1 leads to $\lambda_0 \leqslant 4$; whereas y_2 leads to $\lambda_0 \leqslant \frac{1}{3}\pi^2 + \frac{1}{2}$, which is a better bound. A further improvement can be obtained by using a trial function $y_3 = ay_1 + by_2$ and minimising the Rayleigh quotient with respect to both a and b (you are not required to do this!).

12 Putting the differential equation into standard Sturm–Liouville form (with weight function $w(x) = f(x)$) shows that the given ratio is simply the Rayleigh quotient. To estimate the frequency of the fundamental mode, try a suitable trial function: $y = \sin x$ is appropriate and leads to an approximation $\omega_0 \approx 1/\sqrt{1 + 8/3\pi}$.

13 Since $\mathscr{L}\psi_0 = 0$, the eigenvalue is $\lambda_0 = 0$. The Rayleigh quotient is given by

$$\Lambda[\psi] = \frac{\int_{-\infty}^{\infty}\{\psi'^2 + (x^2-1)\psi^2\}\,\mathrm{d}x}{\int_{-\infty}^{\infty}\psi^2\,\mathrm{d}x}$$

and for the given trial function we obtain $\Lambda[\widetilde{\psi}_0] = \frac{5}{2}a^{-2} + \frac{1}{7}a^2 - 1$. Minimising with respect to a gives the required value of $\widetilde{\lambda}_0$ when $a = \sqrt[4]{35/2}$. Note that $\widetilde{\lambda}_0 - \lambda_0$ is strictly positive, because it is always non-negative (λ_0 is the lowest eigenvalue) but could only be zero if $\widetilde{\psi}_0$ were identically equal to ψ_0.

Comments on or corrections to this problem sheet are very welcome and may be sent to me at `reh10@damtp.cam.ac.uk`.