Pigeonhole I15

Mathematical Methods II — Dr R. E. Hunt
Natural Sciences Tripos Part IB — Lent 2002

Example Sheet 4: Complex Analysis, Contour Integration and Transform Theory

1 The real parts of three analytic functions are

$$\sin x \cosh y; \qquad e^{y^2-x^2}\cos 2xy; \qquad \frac{x}{x^2+y^2}$$

respectively. Use the Cauchy–Riemann relations to find their imaginary parts (up to arbitrary constants) and hence deduce the forms of the complex functions.

2 Show that the real and imaginary parts of an analytic function satisfy Laplace's equation. Verify that the real part of $-E(z - a^2/z)$ is the electrostatic potential for the problem of an earthed conducting circular cylinder of radius a centred at the origin and placed in an external uniform electric field of strength E in the positive x-direction.

3 Show that the two-dimensional function Φ defined by

$$\Phi(x,y) = \operatorname{Im}\left\{\frac{2}{\pi}\log\tanh z\right\},$$

where $z = x + iy$, satisfies Laplace's equation in $x > 0$; and that $\Phi = 0$ on both $y = 0$ and $y = \frac{\pi}{2}$, while on $x = 0$, $\Phi = 1$ for $0 < y < \frac{\pi}{2}$. Deduce the steady-state temperature distribution in a semi-infinite two-dimensional bar of width L, with the (infinitely) long sides held at zero temperature and the short side held at temperature T_0.

4 Where are the zeros and singularities of the following complex functions? Give the orders of the zeros, and classify the singularities.

$$\frac{(z-i)^2}{z+1}; \quad \frac{1}{1+z} - \frac{1}{1-z}; \quad \frac{1}{z^2+i}; \quad \sec^2 \pi z; \quad \sin z^{-2}; \quad \sinh\frac{z}{z^2-1}; \quad \frac{\tanh z}{z}.$$

5 Establish the following general methods for calculating residues. [*Note: These are all very useful in practice, and the student is advised to memorise them.*]

(i) If $f(z)$ has a simple pole, then the residue of $f(z)$ at $z = z_0$ is $\lim_{z\to z_0}\{(z-z_0)f(z)\}$.

(ii) If $f(z)$ is analytic, then the residue of $f(z)/(z - z_0)$ at $z = z_0$ is $f(z_0)$.

(iii) If $1/f(z)$ has a simple pole at $z = z_0$, then its residue at $z = z_0$ is $1/f'(z_0)$.

(iv) If $h(z)$ has a simple zero at $z = z_0$ and $g(z)$ is analytic and non-zero, the residue of $g(z)/h(z)$ at $z = z_0$ is $g(z_0)/h'(z_0)$.

(v) If $f(z)$ has a pole of order N at $z = z_0$, then the residue of $f(z)$ at $z = z_0$ is

$$\lim_{z\to z_0}\left\{\frac{1}{(N-1)!}\frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}}\left((z-z_0)^N f(z)\right)\right\}.$$

6 Find the poles of the following functions and calculate the residues at each pole:

$$\frac{z+1}{z^2}; \qquad \frac{e^{-z}}{z^3}; \qquad \frac{\sin^2 z}{z^5}; \qquad \cot z; \qquad \frac{z^2}{(1+z^2)^2}.$$

7 Sketch possible arrangements of branch cuts for the following, giving the values on either side of each cut:

$$(z^2+1)^{1/2}; \qquad (z^2+1)^{1/3}; \qquad \log\left(\frac{z-i}{z+i}\right)^2.$$

8 (i) State and prove Cauchy's Theorem.

(ii) Suppose that the simple contour C encloses $z = z_0$ in a positive sense and that f is an analytic function. Show that

$$\oint_C (z-z_0)^n \, \mathrm{d}z = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{if } n \text{ is any other integer} \end{cases}$$

and

$$\oint_C \frac{f'(z)\,\mathrm{d}z}{z-z_0} = \oint_C \frac{f(z)\,\mathrm{d}z}{(z-z_0)^2}.$$

9 Suppose that $f(z)$ is analytic in and on the circle $|z-z_0| = r$. Show that for $n \geqslant 0$,

$$\left|f^{(n)}(z_0)\right| \leqslant \frac{n!}{r^n} \max_{|z-z_0|=r} |f(z)|.$$

Hence prove *Liouville's Theorem*: if $f(z)$ is analytic and bounded for all z then f is a constant.

10 Describe the method of the calculus of residues.

By integrating the function $z^n(z-a)^{-1}(z-a^{-1})^{-1}$ around the unit circle in the z-plane (where a is real, $a > 1$, and n is a non-negative integer), evaluate

$$\int_0^{2\pi} \frac{\cos n\theta}{1 - 2a\cos\theta + a^2} \, \mathrm{d}\theta.$$

11 By considering the integral $\oint (z^2+1)^{-1} e^{ikz} \, \mathrm{d}z$ taken around a large semicircle, show that for real positive k,

$$\int_{-\infty}^{\infty} \frac{\cos kx}{x^2+1} \, \mathrm{d}x = \pi e^{-k}.$$

What is the value for $k \leqslant 0$?

12 Verify the following results, where a is a non-zero real constant.

(i) $\displaystyle\int_0^\infty \frac{x^{-a}\,\mathrm{d}x}{x+1} = \frac{\pi}{\sin \pi a} \qquad (0 < a < 1).$

(ii) $\displaystyle\int_0^\pi \frac{a\,\mathrm{d}\theta}{a^2+\sin^2\theta} = \frac{\pi}{\sqrt{1+a^2}} \qquad (a > 1).$

(iii) $\displaystyle\int_0^\infty \frac{x^4}{1+x^8}\,\mathrm{d}x = \frac{\pi}{4}\sqrt{1-1/\sqrt{2}}\,.$

[*Although this can be done with the standard semicircle, you might like to consider instead using a sector of a circle of angle* $\pi/4$.]

(iv) $\displaystyle\int_0^\infty \cos(\tfrac{1}{2}ax^2)\,\mathrm{d}x = \sqrt{\frac{\pi}{4|a|}}\,.$

[*Hint: in this case you must use a sector of a circle.*]

(v) $\displaystyle\int_0^\infty \frac{(\log x)^2\,\mathrm{d}x}{1+x^2} = \frac{\pi^3}{8}\,.$

[*Hint: use a semicircular contour with an appropriate branch cut.*]

13 Find the function whose Fourier transform is $(1+k^4)^{-1}$.

14 By integrating round a rectangular contour with vertices at $\pm R$ and $i\pi \pm R$, where R is a large real constant, or otherwise, show that $\int_0^\infty \operatorname{sech} x\,\mathrm{d}x = \pi/2$.

15 (i) Find the Laplace transforms of the following functions (defined for $t \geqslant 0$), where α, a and b are positive real constants and n is a non-negative integer:

$$e^{\alpha t}; \qquad t^n; \qquad \cosh \alpha t; \qquad t \sin \alpha t; \qquad \begin{cases} e^t & t \geqslant 2, \\ 0 & t < 2; \end{cases} \qquad \begin{cases} 1 & a \leqslant t \leqslant b, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) State the form of the Bromwich integral for the inverse of a Laplace transform, explaining carefully the path of integration used. Verify your answers for part (i) by performing the inverse transformations.

16 Use Laplace transforms to solve the following problems for the function $x(t)$:

(i) $\ddot{x} + 4x = 12t$, with initial conditions $x = 0$, $\dot{x} = 7$ at $t = 0$;

(ii) $\ddot{x} + t\dot{x} - x = 0$, with initial conditions $x = 0$, $\dot{x} = 1$ at $t = 0$.

17 Use Laplace transforms to solve the coupled differential equations

$$\begin{aligned} \dot{x} + \ x + \ y &= f(t) \\ \dot{y} - 3x + 5y &= 0 \end{aligned}$$

with initial conditions $x = y = 0$ at $t = 0$, where $f(t)$ is given by

$$f(t) = \begin{cases} 0 & t < 1, \\ e^{-t} & t \geqslant 1. \end{cases}$$

18 A certain physical system started from rest but then subjected to forcing $f(t)$ satisfies the equation

$$\frac{\mathrm{d}^4 x}{\mathrm{d}t^4} - x = f(t)$$

where $f(t)$, $x(t)$ and its first three derivatives vanish for $t \leqslant 0$. Using the convolution theorem for Laplace transforms, show that the solution can be written as $x = f * g$ where $g(t)$ has transform

$$\bar{g}(p) = \frac{1}{p^4 - 1}.$$

Deduce that

$$x(t) = \tfrac{1}{2} \int_0^t \big(\sinh(t - \tau) - \sin(t - \tau)\big) f(\tau)\, \mathrm{d}\tau,$$

and show that $g(t)$ is the response of the system to forcing $\delta(t)$.

Comments on or corrections to this problem sheet are very welcome and may be sent to me at `reh10@damtp.cam.ac.uk`.