Pigeonhole I9

Mathematical Methods II — Dr R. E. Hunt
Natural Sciences Tripos Part IB — Lent 2002

Example Sheet 1: Variational Methods

1 A coal box, in the shape of a cuboid, is to be placed flush against an outside wall, so that only its top, front and two sides are visible. The owner wishes the box to contain at least a certain volume V of coal, but also wishes to minimise the visible surface area in order to avoid the box becoming an eyesore. What lengths should be chosen for the sides?

2 The temperature within *and* on the surface of a sphere of unit radius is given by $T(x, y, z) = x(y + z)$. Find the minimum and maximum temperature.

3 Find the geodesics on a cylinder of radius a.

4 The new Mayor of London is devoted to schemes for energy saving and wishes to design a fuel-less tube transport system driven by gravity. He proposes that vehicles should travel in frictionless underground tunnels, being dropped from rest at their point of departure A (Waterloo), and then allowed to run freely until they arrive at their destination B (Paddington), a distance L apart at the same level. Show that, neglecting variations in gravity, the minimum travel time is $\sqrt{2\pi L/g}$, and that the tunnels at the end-points should be vertical. (Which should make the tube journey that bit more exciting.)

5 State Fermat's principle governing the paths traced by light rays and explain the conditions under which it applies. Given that in a horizontally stratified medium the refractive index is given by $\mu(z) = \sqrt{a - bz}$, where z is the height and a and b are positive constants, prove that light rays travelling in a vertical plane follow inverted parabolas. Show further that *all* such parabolas have their directrix in the plane $z = a/b$. [*The directrix of a parabola in standard form,* $y^2 = 4ax$, *is the line* $x = -a$.]

6 A particle of unit mass moves in a plane with polar coordinates (r, θ), under the influence of a central force derived from a potential $V(r)$. Write down the action functional for this problem and use Hamilton's principle to find differential equations for $r(t)$ and $\theta(t)$. Give a physical interpretation of these equations. Given that the particle's trajectory is $r = a \sin\theta$ for some constant a, deduce that (up to an arbitrary additive constant) $V \propto r^{-4}$.

7 If $\mu(\mathbf{r}) = |\nabla f(\mathbf{r})|$ for some function f, show that $\int_A^B \mu \, \mathrm{d}l$ between two points A and B is at least $f(B) - f(A)$, with equality if and only if the path of integration lies orthogonal to the family of surfaces $f =$ constant. Deduce that such orthogonal trajectories satisfy Fermat's principle.

8 A soap film is bounded by two circular wires at $r = a$, $z = \pm b$ in cylindrical polar coordinates (r, θ, z). Assuming that the soap surface is cylindrically symmetric, show that the equation of the surface of minimal area is

$$r = c\cosh(z/c)$$

where c satisfies the condition $a/c = \cosh(b/c)$. Show graphically that this condition has no solution for c if b/a is larger than a certain critical ratio. What happens to the soap surface as b/a is increased from below this ratio to above it?

9 An area is enclosed by joining two fixed points a distance a apart on a straight wall with a given length l of flexible fencing ($a < l < \pi a$). How is the area maximised?

10 Show from first principles that the equivalent of Euler's equation for the function $x(t)$ which extremises the integral

$$\int_{t_1}^{t_2} f(t, x, \dot{x}, \ddot{x})\,\mathrm{d}t$$

with fixed values of both $x(t)$ and $\dot{x}(t)$ at $t = t_1$ and t_2 is

$$\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial f}{\partial \ddot{x}}\right) = 0.$$

Hence find the function $x(t)$ with $x(1) = 1$, $\dot{x}(1) = -2$, $x(2) = \frac{1}{4}$ and $\dot{x}(2) = -\frac{1}{4}$ that minimises $\int_1^2 t^4\{\ddot{x}(t)\}^2\,\mathrm{d}t$.

11 Consider the Sturm–Liouville problem

$$-(1 + x^2)y'' - 2xy' = \lambda y$$

with $y(\pm 1) = 0$. Use the Rayleigh–Ritz method to obtain an upper bound on the lowest eigenvalue by using the trial function $y_1 = 1 - x^2$. Show that a better bound is obtained from the trial function $y_2 = \cos(\pi x/2)$ and explain how a further improvement could be achieved by considering y_1 and y_2 in combination. [$\int_{-1}^{1} x^2 \sin^2(\pi x/2)\,dx = \frac{1}{3} + \frac{2}{\pi^2}$.]

12 The differential equation governing small transverse displacements $y(x)$ of a string with fixed end-points at $x = 0$ and $x = \pi$ is

$$y'' + \omega^2 f(x) y = 0$$

where ω is the angular frequency of the vibration and f is a positive function. Show that the allowed values of ω^2 are given by the stationary values of

$$\frac{\int_0^\pi y'^2\,\mathrm{d}x}{\int_0^\pi f(x)y^2\,\mathrm{d}x}.$$

Use this fact to find an approximate value for the angular frequency of the fundamental mode when $f(x) = 1 + \sin x$.

13 Show that $\psi_0 = \exp(-\frac{1}{2}x^2)$ is an eigenfunction of the operator

$$\mathscr{L} = -\frac{d^2}{dx^2} + (x^2 - 1)$$

acting on functions $\psi(x)$ for which $\psi \to 0$ as $|x| \to \infty$, and find the corresponding eigenvalue λ_0. This is in fact the lowest eigenvalue of the problem.

Use the Rayleigh–Ritz method with trial function

$$\widetilde{\psi}_0 = \begin{cases} b(a^2 - x^2) & |x| < a \\ 0 & |x| \geqslant a \end{cases}$$

where a and b are adjustable constants, to obtain the approximation

$$\widetilde{\lambda}_0 = \sqrt{10/7} - 1$$

to λ_0. Comment on the sign of $\widetilde{\lambda}_0 - \lambda_0$.

Comments on or corrections to this problem sheet are very welcome and may be sent to me at `reh10@damtp.cam.ac.uk`.