Worked Example
Electrostatics: Using the Integral Solution of Poisson's Equation

Consider a wire of length $2L$ carrying a charge density μ per unit length, lying along the z-axis from $z = -L$ to $+L$. What is the electric potential Φ?

The charge distribution is $\rho(\mathbf{x}) = \mu\delta(x)\delta(y)$ for $-L \leqslant z \leqslant L$ (and zero for $|z| > L$). We shall use the integral solution of Poisson's equation in the whole of space to obtain the potential at a point (x_0, y_0, z_0). We need Green's function, which is simply the fundamental solution here.

$$\begin{aligned}
\Phi(\mathbf{x}_0) &= \iiint\limits_{\mathbb{R}^3} \frac{\rho(\mathbf{x})}{4\pi\epsilon_0|\mathbf{x} - \mathbf{x}_0|}\,\mathrm{d}V \\
&= \int_{-L}^{L} \frac{\mu}{4\pi\epsilon_0|(0,0,z) - \mathbf{x}_0|}\,\mathrm{d}z \\
&= \frac{\mu}{4\pi\epsilon_0}\int_{-L}^{L} \frac{\mathrm{d}z}{\sqrt{x_0^2 + y_0^2 + (z - z_0)^2}} \\
&= \frac{\mu}{4\pi\epsilon_0}\left[\sinh^{-1}\frac{z - z_0}{\sqrt{x_0^2 + y_0^2}}\right]_{-L}^{L} \\
&= \frac{\mu}{4\pi\epsilon_0}\left\{\sinh^{-1}\frac{L - z_0}{\sqrt{x_0^2 + y_0^2}} + \sinh^{-1}\frac{L + z_0}{\sqrt{x_0^2 + y_0^2}}\right\}.
\end{aligned}$$

This is true for arbitrary locations $\mathbf{x}_0$, so replacing $\mathbf{x}_0$ by $\mathbf{x}$ we obtain

$$\Phi(x, y, z) = \frac{\mu}{4\pi\epsilon_0}\left\{\sinh^{-1}\frac{L - z}{\sqrt{x^2 + y^2}} + \sinh^{-1}\frac{L + z}{\sqrt{x^2 + y^2}}\right\}.$$

In particular, the potential at a point in the (x, y)-plane is given by

$$\Phi(x, y, 0) = \frac{\mu}{2\pi\epsilon_0}\sinh^{-1}\bigl(L/\sqrt{x^2 + y^2}\bigr).$$

Note, for completeness, that for very large L, i.e., in the limit as $L \to \infty$, it is possible to check (using $\sinh^{-1} x \sim \ln x$ as $x \to \infty$) that

$$\Phi \to -\frac{\mu}{2\pi\epsilon_0}\ln\sqrt{x^2 + y^2} + \text{constant},$$

which verifies an earlier result we obtained for the two-dimensional field around an infinitely long wire.