Worked Example
Diffusion of a Solute past a Solid Sphere

Consider fluid at rest surrounding a fixed solid sphere of radius a at the origin. The fluid contains a solute which diffuses through the fluid, and we are interested in the steady state. At large distances from the sphere (where the sphere has negligible effect) we assume that there is a constant flux of solute parallel to the z-axis of magnitude F (possibly due, for example, to an externally imposed concentration gradient).

The flux is $-k\nabla\Phi$ where Φ is the concentration. There can be no flux across $r = a$, so $\hat{\mathbf{e}}_r \cdot \nabla\Phi = 0$ on $r = a$, or equivalently $\frac{\partial\Phi}{\partial r}(a, \theta) = 0$ for all θ.

Far from the sphere, we must have $\nabla\Phi \sim -\frac{F}{k}\hat{\mathbf{e}}_z$, i.e., $\Phi \sim -\frac{F}{k}z$; so we require that as $r \to \infty$, $\Phi \sim -\frac{F}{k}r\cos\theta = -\frac{F}{k}rP_1(\cos\theta)$.

We use the general axisymmetric solution, and must choose the arbitrary constants to ensure the correct behaviour as $r \to \infty$. This can only occur if $A_1 = -\frac{F}{k}$ and $A_n = 0$ for all $n \geqslant 2$. Thus

$$\Phi = A_0 - \frac{F}{k}rP_1(\cos\theta) + \sum_{n=0}^{\infty} B_n r^{-n-1} P_n(\cos\theta).$$

On $r = a$, we must have

$$\left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = -\frac{F}{k}P_1(\cos\theta) - \sum_{n=0}^{\infty}(n+1)B_n a^{-n-2}P_n(\cos\theta) = 0$$

for all θ. Using the orthogonality of Legendre polynomials (multiply by $P_m(\cos\theta)$, substitute $\zeta = \cos\theta$, and integrate from $\zeta = -1$ to 1), or by inspection, we find that $B_0 = 0$, $B_1 = -Fa^3/2k$ and $B_n = 0$ for all $n \geqslant 2$. So the solution is

$$\Phi = A_0 - \frac{F}{k}\left(r + \frac{a^3}{2r^2}\right)\cos\theta,$$

and A_0 remains an arbitrary constant (it measures, in some sense, the average of the concentrations far up and downstream).

Note that the boundary conditions involved only $P_1(\cos\theta)$ and no other P_n; and so does the solution. This is usual: boundary conditions can often be expressed in terms of just a few P_n, and only those terms need be retained from the general solution. For this purpose it is useful to know the following:

$$\begin{aligned} 1 &= P_0(\cos\theta) \\ \cos\theta &= P_1(\cos\theta) \\ \cos^2\theta &= \tfrac{2}{3}P_2(\cos\theta) + \tfrac{1}{3}P_0(\cos\theta) \end{aligned}$$