Worked Example
Steady-State Temperature Distribution in a Cylinder

An infinitely long cylinder of radius a is heated on its boundary as shown. The steady-state temperature $T(r,\theta)$ (note no dependence on z) satisfies

$$\nabla^2 T = 0 \qquad \text{in } r < a$$

subject to

$$T(a,\theta) = \begin{cases} +T_0 & 0 \leqslant \theta < \pi, \\ -T_0 & \pi \leqslant \theta < 2\pi. \end{cases}$$

The general solution for plane polar coordinates applies; we choose to use it in its second form as given in the lecture notes. We require that the temperature be finite at $r = 0$ for a physically realistic solution: so $C_0 = 0$, and also, for all *negative* n, $A_n = B_n = 0$ (since they are the coefficients of $r^n \left\{ \begin{smallmatrix} \cos \\ \sin \end{smallmatrix} \right\} n\theta$). Finally, T must be periodic in θ (i.e., not multi-valued), so $B_0 = 0$. Hence

$$T(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta).$$

On $r = a$ this gives

$$T(a,\theta) = A_0 + \sum_{n=1}^{\infty} (A_n a^n \cos n\theta + B_n a^n \sin n\theta).$$

This is a standard Fourier series, so we may calculate the Fourier coefficients using the standard formulae:

$$\begin{aligned}
A_0 &= \frac{1}{2\pi} \int_0^{2\pi} T(a,\theta)\,\mathrm{d}\theta = 0 \qquad \text{(by anti-symmetry of } T(a,\theta)\text{)} \\
A_n a^n &= \frac{1}{\pi} \int_0^{2\pi} T(a,\theta) \cos n\theta \,\mathrm{d}\theta = 0 \\
B_n a^n &= \frac{1}{\pi} \int_0^{2\pi} T(a,\theta) \sin n\theta \,\mathrm{d}\theta \\
&= \frac{1}{\pi} \int_0^{\pi} T_0 \sin n\theta \,\mathrm{d}\theta - \frac{1}{\pi} \int_{\pi}^{2\pi} T_0 \sin n\theta \,\mathrm{d}\theta \\
&= \begin{cases} 4T_0/n\pi & n \text{ odd}, \\ 0 & n \text{ even}. \end{cases}
\end{aligned}$$

Hence the final solution for all r and θ is

$$T = \frac{4T_0}{\pi} \sum_{n \text{ odd}} \frac{r^n}{n a^n} \sin n\theta.$$