Worked Example
The Catenary

Consider a uniform chain of length L, with mass per unit length ρ, hanging under gravity between the points $(-1, 1)$ and $(1, 1)$. It adopts a form of minimum potential energy, that is it minimises

$$\int_{-1}^{1} \rho g y \, \mathrm{d}l \propto F[y] \equiv \int_{-1}^{1} y\sqrt{1 + y'^2} \, \mathrm{d}x$$

subject to the prescribed length,

$$L = G[y] \equiv \int_{-1}^{1} \sqrt{1 + y'^2} \, \mathrm{d}x.$$

This is equivalent to minimising $F - \lambda G$, i.e., to solving

$$\delta \int_{-1}^{1} (y - \lambda)\sqrt{1 + y'^2} \, \mathrm{d}x = 0.$$

The integrand has no explicit x-dependence, so we use the first integral

$$\begin{aligned} c &= (y - \lambda)\sqrt{1 + y'^2} - y'(y - \lambda)\frac{y'}{\sqrt{1 + y'^2}} \\ &= \frac{y - \lambda}{\sqrt{1 + y'^2}}, \end{aligned}$$

where c is a constant, whence

$$x = \int \frac{c \, \mathrm{d}y}{\sqrt{(y - \lambda)^2 - c^2}}.$$

Making the substitution $y = \lambda + c\cosh\theta$ we obtain

$$x = c\cosh^{-1}\left(\frac{y - \lambda}{c}\right) + x_0$$

where x_0 is an arbitrary constant of integration. Hence the solution is

$$y = \lambda + c\cosh\left(\frac{x - x_0}{c}\right),$$

which is a *catenary*.

We have three unknown constants, to be found using the equation for y at each of the two end-points, together with the constraint equation. We immediately obtain $x_0 = 0$ by symmetry (or by solving the end-point equations for x_0). Now $y' = \sinh(x/c)$ and hence $\sqrt{1 + y'^2} = \cosh(x/c)$; so

$$\begin{aligned} L &= \int_{-1}^{1} \cosh\frac{x}{c} \, \mathrm{d}x \\ &= 2c\sinh\frac{1}{c}. \end{aligned}$$

This equation must, in general, be solved numerically for c given L; then λ can be found using the end-point at $(1, 1)$,

$$1 = \lambda + c \cosh \frac{1}{c}.$$

This completes the solution.