# Summary of Results from Chapter 5: Contour Integration and Transform Theory

## *Cauchy's Theorem*

If $f(z)$ is analytic in a simply-connected domain $R$, then for any simple closed curve $C$ in $R$,

$$\oint_C f(z)\,\mathrm{d}z = 0.$$

We deduce that for any integral between two points $z_0$ and $z_1$, or round a closed curve, we may deform one contour of integration into another without affecting the value of the integral so long as we do not cross any singularities of the integrand during the deformation.

## *The Integral of f′(z)*

If $f(z)$ is analytic in a simply-connected domain $R$ and the contour of integration lies entirely in $R$, then

$$\int_{z_0}^{z_1} f'(z)\,\mathrm{d}z = f(z_1) - f(z_0).$$

## *The Residue Theorem*

If $f(z)$ is analytic in a simply-connected domain $R$ except for a finite number of poles at $z = z_1, z_2, \ldots, z_n$, and $C$ is a simple closed anticlockwise contour in $R$ encircling the poles, then

$$\oint_C f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \operatorname*{res}_{z=z_k} f(z).$$

In particular, if $C$ is a simple closed curve encircling $z_0$ in a positive (anticlockwise) sense and $n$ is an integer,

$$\oint_C (z-z_0)^n\,\mathrm{d}z = \begin{cases} 0 & n \neq -1, \\ 2\pi i & n = -1. \end{cases}$$

## *Cauchy's Formula*

If $f(z)$ is analytic in a simply-connected domain $R$ and $z_0$ lies in $R$, then

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,\mathrm{d}z$$

for any simple closed anticlockwise contour $C$ in $R$ encircling $z_0$.

If instead $f(z)$ has a singularity at $z_0$, with Laurent expansion $\sum\limits_{n=-\infty}^{\infty} a_n(z-z_0)^n$, then the coefficients of the expansion are given by

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z.$$

## Jordan's Lemma

Let $C_R$ be the semicircular contour of radius $R$ in the upper half plane with centre at the origin, traversed from $+R$ on the real axis to $-R$; let $C'_R$ be the semicircular contour of radius $R$ in the lower half plane with centre at the origin, traversed from $+R$ to $-R$; and let $f(z)$ be an analytic function (except possibly for a finite number of poles) which satisfies $f(z) \to 0$ as $|z| \to \infty$. Then for any real constant $\lambda > 0$,

$$\int_{C_R} f(z)e^{i\lambda z}\,\mathrm{d}z \to 0$$

as $R \to \infty$; while for $\lambda < 0$,

$$\int_{C'_R} f(z)e^{i\lambda z}\,\mathrm{d}z \to 0$$

as $R \to \infty$.

## Laplace Transforms

The Laplace transform of a function $f(t)$ which vanishes for $t < 0$ is

$$\mathscr{L}[f(t)] \equiv \bar{f}(p) = \int_0^\infty f(t)e^{-pt}\,\mathrm{d}t.$$

The Laplace transform operator satisfies

$$\mathscr{L}[tf(t)] = -\frac{\mathrm{d}}{\mathrm{d}p}\bar{f}(p)$$

and

$$\mathscr{L}\left[\frac{\mathrm{d}f}{\mathrm{d}t}\right] = p\bar{f}(p) - f(0).$$

The convolution of two functions $f$ and $g$ which vanish for $t < 0$ is

$$(f * g)(t) = \int_0^t f(t-t')g(t')\,\mathrm{d}t'$$

and its Laplace transform is

$$\mathscr{L}[f * g] = \bar{f}(p)\bar{g}(p).$$

The Bromwich inversion formula is

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \bar{f}(p)e^{pt}\,\mathrm{d}p$$

where the Bromwich inversion contour runs along the line $\mathrm{Re}\,p = \gamma$, where $\gamma$ is a real constant which lies to the right of all the singularities of $\bar{f}(p)$.

If $\bar{f}(p) \to 0$ as $|p| \to \infty$, and if $\bar{f}(p)$ has poles at $p = p_1, \ldots, p_n$ (but no other singularities, e.g., branch cuts), then

$$f(t) = \begin{cases} 0 & t < 0, \\ \displaystyle\sum_{k=1}^{n} \operatorname*{res}_{p=p_k}\left(\bar{f}(p)e^{pt}\right) & t > 0. \end{cases}$$