Worked Example
Solving Differential Equations using the Laplace Transform and its Inverse

We shall solve

$$\ddot{x} + x = 2\sin t$$

for $x(t)$, with initial conditions $x(0) = 0$, $\dot{x}(0) = 2$. Taking the Laplace transform with respect to time,

$$\big(p^2\bar{x}(p) - px(0) - \dot{x}(0)\big) + \bar{x}(p) = \frac{2}{p^2+1}.$$

Using the initial conditions, we obtain

$$p^2\bar{x} - 2 + \bar{x} = \frac{2}{p^2+1}$$

from which we deduce that

$$\bar{x} = \frac{2p^2+4}{(p^2+1)^2}.$$

To invert this we write down the Bromwich inversion formula

$$x(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{2p^2+4}{(p^2+1)^2}\, e^{pt}\,\mathrm{d}p.$$

The integrand has poles of order two at $p = \pm i$, so we must have $\gamma > 0$ in order that the integration contour lies to the right of the singularities.

What are the residues at the poles? At $p = i$, the residue is

$$\begin{aligned}\lim_{p\to i}\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{2p^2+4}{(p+i)^2}\,e^{pt}\right) &= \lim_{p\to i}\left(\frac{(p+i)\big(4p+(2p^2+4)t\big) - 2(2p^2+4)(p+i)}{(p+i)^3}\,e^{pt}\right)\\ &= -\tfrac{1}{2}(t+3i)e^{it}.\end{aligned}$$

Similarly, at $p = -i$ the residue is $-\frac{1}{2}(t-3i)e^{-it}$.

As $|p| \to \infty$, $\bar{x}(p) = O(|p|^{-2}) \to 0$; hence for $t > 0$ we close the integration contour to the left, picking up the residues from the poles to obtain

$$\begin{aligned}x(t) &= -\tfrac{1}{2}(t+3i)e^{it} - \tfrac{1}{2}(t-3i)e^{-it}\\ &= -\tfrac{1}{2}\big(2t\cos t + 3i(2i\sin t)\big)\\ &= 3\sin t - t\cos t.\end{aligned}$$