Worked Example
Contour Integration: Singular Point on the Real Axis

We wish to evaluate

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x.$$

This integrand is well-behaved at the origin, so the integral is non-singular. But the obvious approach via contour integration using

$$\frac{1}{2i}\int_{-\infty}^{\infty} \frac{e^{iz}-e^{-iz}}{z}\,\mathrm{d}z$$

runs into trouble because we cannot apply Jordan's Lemma to the integrand. To get round this we might try to split it into two separate integrals, to each of which Jordan's Lemma does apply, but then we find that our contour passes *through* a pole of the integrand.

Instead, we write

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x &= \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \left(\int_{-R}^{-\varepsilon} \frac{\sin x}{x}\,\mathrm{d}x + \int_{\varepsilon}^{R} \frac{\sin x}{x}\,\mathrm{d}x\right)\\ &= \operatorname{Im} \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \left(\int_{-R}^{-\varepsilon} \frac{e^{iz}}{z}\,\mathrm{d}z + \int_{\varepsilon}^{R} \frac{e^{iz}}{z}\,\mathrm{d}z\right).\end{aligned}$$

Let C be the contour from $-R$ to $-\varepsilon$, then round a semicircle C_ε of radius ε, then from ε to R, and returning via a semicircle C_R of radius R. Then C encloses no poles of e^{iz}/z, so

$$\int_{-R}^{-\varepsilon} \frac{e^{iz}}{z}\,\mathrm{d}z + \int_{\varepsilon}^{R} \frac{e^{iz}}{z}\,\mathrm{d}z = -\int_{C_\varepsilon} \frac{e^{iz}}{z}\,\mathrm{d}z - \int_{C_R} \frac{e^{iz}}{z}\,\mathrm{d}z.$$

Jordan's Lemma tells us that the integral round C_R vanishes as $R\to\infty$. On C_ε, $z=\varepsilon e^{i\theta}$ and $e^{iz}=1+O(\varepsilon)$; so

$$\int_{C_\varepsilon} \frac{e^{iz}}{z}\,\mathrm{d}z = \int_{\pi}^{0} \frac{1+O(\varepsilon)}{\varepsilon e^{i\theta}}\, i\varepsilon e^{i\theta}\,\mathrm{d}\theta = -i\pi + O(\varepsilon).$$

Hence, taking the limit as $\varepsilon\to 0$ and $R\to\infty$,

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \operatorname{Im}(i\pi) = \pi.$$

A similar method works for

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,\mathrm{d}x;$$

write $\sin^2 x = \frac{1}{2}\,\mathrm{Re}(1 - e^{2ix})$, and then

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,\mathrm{d}x = \tfrac{1}{2}\,\mathrm{Re}\lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \left(\int_{-R}^{-\varepsilon} \frac{1-e^{2iz}}{z^2}\,\mathrm{d}z + \int_{\varepsilon}^{R} \frac{1-e^{2iz}}{z^2}\,\mathrm{d}z \right)$$

$$= \tfrac{1}{2}\,\mathrm{Re}\lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \left(-\int_{C_\varepsilon} \frac{1-e^{2iz}}{z^2}\,\mathrm{d}z - \int_{C_R} \frac{1-e^{2iz}}{z^2}\,\mathrm{d}z \right).$$

The integral round C_R can be shown to vanish as $R \to \infty$ by standard techniques (Jordan's Lemma is not, however, applicable), and the integral round C_ε can be evaluated as before (expanding e^{2iz} to slightly higher order in ε than before), giving

$$\int_{C_\varepsilon} \frac{1-e^{2iz}}{z^2}\,\mathrm{d}z = -2\pi + O(\varepsilon).$$

Hence

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,\mathrm{d}x = \pi$$

as well!

An alternative approach for both examples is to note that, for instance, $(\sin z)/z$ has a removable singularity at the origin. Having removed the singularity, we have an analytic integrand, and therefore the original contour along the real axis can be moved to one which does not pass through the origin. It is now possible to write $\sin z = (e^{iz} - e^{-iz})/2i$, split the integrand in two, and apply Jordan's Lemma to each part separately.