Worked Example
Contour Integration: Inverse Fourier Transforms

Consider the real function

$$f(x) = \begin{cases} 0 & x < 0 \\ e^{-ax} & x > 0 \end{cases}$$

where $a > 0$ is a real constant. The Fourier Transform of $f(x)$ is

$$\begin{aligned} \widetilde{f}(k) &= \int_{-\infty}^{\infty} f(x) e^{-ikx}\,\mathrm{d}x \\ &= \int_{0}^{\infty} e^{-ax-ikx}\,\mathrm{d}x \\ &= -\frac{1}{a+ik}\left[e^{-ax-ikx}\right]_0^\infty \\ &= \frac{1}{a+ik}. \end{aligned}$$

We shall verify the Inverse Fourier Transform by evaluating

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{f}(k) e^{ikx}\,\mathrm{d}k.$$

In the complex k-plane, let C_0 be the contour from $-R$ to R on the real axis, C_R be the semicircle of radius R in the upper half plane and C'_R be the semicircle of radius R in the lower half plane. Let C be C_0 followed by C_R (this is known as *closing in the upper half plane*), and let C' be C_0 followed by C'_R (*closing in the lower half plane*).

Now $\widetilde{f}(k)$ has only one pole, at $k = ia$, which is simple, so

$$\oint_C \widetilde{f}(k) e^{ikx}\,\mathrm{d}k = 2\pi i \operatorname*{res}_{k=ia} \frac{e^{ikx}}{i(k-ia)} = 2\pi e^{-ax},$$

whereas

$$\oint_{C'} \widetilde{f}(k) e^{ikx}\,\mathrm{d}k = 0.$$

(Note that C' is traversed in a negative sense, so if there had been any poles within C' we would have had to introduce a minus sign.)

Now, if $x > 0$, we can apply Jordan's Lemma (with $\lambda = x$) to C_R to show that $\int_{C_R} \widetilde{f}(k) e^{ikx}\,\mathrm{d}k \to 0$ as $R \to \infty$, since $\widetilde{f}(k) = O(1/k)$ as $|k| \to \infty$. Hence for $x > 0$,

$$\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{f}(k) e^{ikx}\,\mathrm{d}k &= \frac{1}{2\pi}\lim_{R\to\infty}\int_{C_0} \widetilde{f}(k) e^{ikx}\,\mathrm{d}k \\ &= \frac{1}{2\pi}\lim_{R\to\infty}\left(\oint_C \widetilde{f}(k) e^{ikx}\,\mathrm{d}k - \int_{C_R} \widetilde{f}(k) e^{ikx}\,\mathrm{d}k\right) \\ &= e^{-ax}. \end{aligned}$$

For $x < 0$ we close in the lower half plane instead, and the same analysis applies to C':

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{f}(k)e^{ikx}\,\mathrm{d}k = \frac{1}{2\pi}\lim_{R\to\infty}\left(\oint_{C'} \widetilde{f}(k)e^{ikx}\,\mathrm{d}k - \int_{C'_R} \widetilde{f}(k)e^{ikx}\,\mathrm{d}k\right)$$
$$= 0.$$

Hence, combining the above results, we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{f}(k)e^{ikx}\,\mathrm{d}k = \begin{cases} 0 & x < 0 \\ e^{-ax} & x > 0 \end{cases}$$

as expected.

Note that by taking real and imaginary parts of this equality we can deduce the values of particular real integrals. The imaginary part gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{a\sin kx - k\cos kx}{a^2+k^2}\,\mathrm{d}k = 0,$$

which is obvious anyway as the integrand is an odd function of k. But the real part gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{a\cos kx + k\sin kx}{a^2+k^2}\,\mathrm{d}k = \begin{cases} 0 & x < 0 \\ e^{-ax} & x > 0 \end{cases}$$

and in particular

$$\int_{-\infty}^{\infty} \frac{a\cos\theta + \theta\sin\theta}{a^2+\theta^2}\,\mathrm{d}\theta = 2\pi e^{-a}.$$