Worked Example
Contour Integration: Integration Round a Branch Cut

We wish to evaluate

$$I = \int_0^\infty \frac{x^\alpha}{1 + \sqrt{2}\,x + x^2}\,\mathrm{d}x$$

where $-1 < \alpha < 1$ so that the integral converges. We will need a branch cut for z^α; we take this along the positive real axis and define

$$z^\alpha = r^\alpha e^{i\alpha\theta}$$

where $z = re^{i\theta}$ and $0 \leqslant \theta < 2\pi$.

Consider

$$\oint_C \frac{z^\alpha}{1 + \sqrt{2}\,z + z^2}\,\mathrm{d}z$$

where the *keyhole contour* C consists of a large circle C_R of radius R, a small circle C_ε of radius ε (to avoid the singularity of z^α at $z = 0$) and two lines just above and below the branch cut, as shown.

The contribution from C_R is $O(R^{\alpha-2}) \times 2\pi R = O(R^{\alpha-1}) \to 0$ as $R \to \infty$.

The contribution from C_ε is (substituting $z = \varepsilon e^{i\theta}$ on C_ε)

$$\int_{2\pi}^0 \frac{\varepsilon^\alpha e^{i\alpha\theta}}{1 + \sqrt{2}\,\varepsilon e^{i\theta} + \varepsilon^2 e^{2i\theta}}\, i\varepsilon e^{i\theta}\,\mathrm{d}\theta = O(\varepsilon^{\alpha+1}) \to 0$$

as $\varepsilon \to 0$.

The contribution from just above the branch cut is

$$\int_\varepsilon^R \frac{x^\alpha}{1 + \sqrt{2}\,x + x^2}\,\mathrm{d}x \to I$$

as $\varepsilon \to 0$ and $R \to \infty$. The contribution from just below the branch cut is

$$\int_R^\varepsilon \frac{x^\alpha e^{2\alpha\pi i}}{1 + \sqrt{2}\,x + x^2}\,\mathrm{d}x \to -e^{2\alpha\pi i} I$$

as $\varepsilon \to 0$ and $R \to \infty$.

Hence

$$\oint_C \frac{z^\alpha}{1 + \sqrt{2}\,z + z^2}\,\mathrm{d}z \to (1 - e^{2\alpha\pi i})I$$

as $\varepsilon \to 0$ and $R \to \infty$.

But the integrand is equal to

$$\frac{z^\alpha}{(z - e^{3\pi i/4})(z - e^{5\pi i/4})}$$

(by finding the roots of the quadratic), so the poles inside C are at $e^{3\pi i/4}$ with residue $e^{3\alpha\pi i/4}/(\sqrt{2}\,i)$ and at $e^{5\pi i/4}$ with residue $e^{5\alpha\pi i/4}/(-\sqrt{2}\,i)$. Hence, taking the limits $\varepsilon \to 0$ and $R \to \infty$,

$$(1 - e^{2\alpha\pi i})I = 2\pi i \left(\frac{e^{3\alpha\pi i/4}}{\sqrt{2}\,i} + \frac{e^{5\alpha\pi i/4}}{-\sqrt{2}\,i} \right),$$

i.e.,

$$e^{\alpha\pi i}(e^{-\alpha\pi i} - e^{\alpha\pi i})I = \sqrt{2}\,\pi e^{\alpha\pi i}(e^{-\alpha\pi i/4} - e^{\alpha\pi i/4}).$$

We conclude that

$$I = \sqrt{2}\,\pi \, \frac{\sin(\alpha\pi/4)}{\sin(\alpha\pi)}.$$