Worked Example
Geodesics on the Surface of a Sphere

Recall that in orthogonal curvilinear coordinates (q_1, q_2, q_3),

$$\mathrm{d}\mathbf{r} = h_1\,\mathrm{d}q_1\,\mathbf{e}_1 + h_2\,\mathrm{d}q_2\,\mathbf{e}_2 + h_3\,\mathrm{d}q_3\,\mathbf{e}_3.$$

In spherical polar coordinates,

$$\mathrm{d}\mathbf{r} = \mathrm{d}r\,\mathbf{e}_r + r\,\mathrm{d}\theta\,\mathbf{e}_\theta + r\sin\theta\,\mathrm{d}\phi\,\mathbf{e}_\phi.$$

Without loss of generality, we may take the sphere to be of unit radius: the length of a path from A to B is then

$$\begin{aligned} L &= \int_A^B |\mathrm{d}\mathbf{r}| \\ &= \int_A^B \sqrt{\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2} \qquad [\text{since } \mathrm{d}r = 0] \\ &= \int_{\theta_A}^{\theta_B} \sqrt{1 + \sin^2\theta\,\phi'^2}\,\mathrm{d}\theta \end{aligned}$$

where the path is described by the function $\phi(\theta)$. Using Euler's equation,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\partial}{\partial\phi'}\sqrt{1 + \sin^2\theta\,\phi'^2}\right) = \frac{\partial}{\partial\phi}\sqrt{1 + \sin^2\theta\,\phi'^2} = 0$$

so that

$$\frac{\sin^2\theta\,\phi'}{\sqrt{1 + \sin^2\theta\,\phi'^2}}$$

is a constant, c say. Hence

$$\phi' = \frac{c}{\sin\theta\sqrt{\sin^2\theta - c^2}}$$

and the problem reduces to integrating this with respect to θ.

Substitute $u = \cot\theta$ so that $\mathrm{d}u = -\operatorname{cosec}^2\theta\,\mathrm{d}\theta$. Then

$$\begin{aligned} \phi &= \int \frac{-c\,\mathrm{d}u}{\sqrt{1 - c^2\operatorname{cosec}^2\theta}} \\ &= \int \frac{-c\,\mathrm{d}u}{\sqrt{1 - c^2(1 + u^2)}} \\ &= \int \frac{-\mathrm{d}u}{\sqrt{a^2 - u^2}} \qquad \text{where } a = \frac{\sqrt{1 - c^2}}{c} \\ &= \cos^{-1}(u/a) + \phi_0 \end{aligned}$$

where ϕ_0 is a constant of integration. Hence the geodesic path is given by

$$\cot\theta = a\cos(\phi - \phi_0)$$

and the arbitrary constants a and ϕ_0 must be found using the end-points. This is a *great circle path.*