Worked Example
Contour Integration: Integrals of Trigonometric Functions

We wish to evaluate

$$I = \int_0^{2\pi} \frac{\mathrm{d}\theta}{a + \cos\theta}$$

where $a > 1$ (so that the integrand is always finite). Substitute $z = e^{i\theta}$, so that $\mathrm{d}z = iz\,\mathrm{d}\theta$ and $\cos\theta = \frac{1}{2}(z + z^{-1})$. As θ increases from 0 to 2π, z moves round the circle C of radius 1 in the complex plane. Hence

$$I = \oint_C \frac{(iz)^{-1}\,\mathrm{d}z}{a + \frac{1}{2}(z + z^{-1})} = -2i \oint_C \frac{\mathrm{d}z}{z^2 + 2az + 1}.$$

The integrand has poles at

$$z_\pm = -a \pm \sqrt{a^2 - 1},$$

both on the real axis. Note that z_+ is inside the unit circle (check that $a - 1 < \sqrt{a^2 - 1} < a$, so $-1 < z_+ < 0$) whereas z_- is outside it. The integrand is equal to

$$\frac{1}{(z - z_+)(z - z_-)}$$

so the residue at $z = z_+$ is $1/(z_+ - z_-) = 1/2\sqrt{a^2 - 1}$. Hence

$$I = -2i \left(\frac{2\pi i}{2\sqrt{a^2 - 1}} \right) = \frac{2\pi}{\sqrt{a^2 - 1}}.$$