Summary of Results from Chapter 4: Complex Analysis

Analyticity

A function $f(z)$ is differentiable at z if the limit

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

exists and is independent of the direction taken by δz in the limiting process.

A function $f(z)$ is analytic (or regular) in a region $R \subseteq \mathbb{C}$ if $f'(z)$ exists and is continuous for all $z \in R$. It is analytic at a point z_0 if it is analytic in some neighbourhood of z_0, i.e., in some region enclosing z_0.

$f(z)$ is differentiable at z_0 if and only if the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

hold at z_0, where

$$u(x, y) = \operatorname{Re} f(x + iy) \qquad \text{and} \qquad v(x, y) = \operatorname{Im} f(x + iy).$$

The functions u and v are harmonic, i.e., satisfy Laplace's equation in two dimensions.

Laurent Expansions

If $f(z)$ is analytic in some annulus centred at z_0 then there exist complex constants a_n such that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

within the annulus.

An isolated singularity of a function $f(z)$ is a point z_0 at which f is singular, but where otherwise f is analytic within some neighbourhood of z_0. A Laurent expansion of f always exists about an isolated singularity.

If there are non-zero a_n for arbitrarily large negative n then f has an essential isolated singularity at z_0.

If $a_n = 0$ for all $n < -N$, but $a_{-N} \neq 0$, where N is a positive integer, then f has a pole of order N at z_0.

If $a_n = 0$ for all $n < 0$ then f has a removable singularity at z_0.

Residues

If $f(z)$ has an isolated singularity at z_0, then its residue there is given by the coefficient a_{-1} in the Laurent expansion.

If $f(z)$ has a pole of order N at z_0, then

$$\operatorname*{res}_{z=z_0} f(z) = \lim_{z\to z_0}\left\{\frac{1}{(N-1)!}\frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}}\big((z-z_0)^N f(z)\big)\right\}.$$

In particular, if $f(z)$ has a simple pole at z_0 then

$$\operatorname*{res}_{z=z_0} f(z) = \lim_{z\to z_0}\{(z-z_0)f(z)\}.$$

The residue of $f(z)$ at infinity is defined to be equal to the residue of $f(1/\zeta)$ at $\zeta = 0$.

Branch Points

A branch point of a function is a point which it is impossible to encircle with a curve upon which the function is continuous and single-valued. The function has a branch point singularity there.

Branch Cuts

The canonical branch cut for both $\log z$ and z^α, where α is not an integer, is along the negative real axis from 0 to $-\infty$. With this cut,

$$\log z = \log r + i\theta$$

and

$$z^\alpha = r^\alpha e^{i\alpha\theta}$$

where $z = re^{i\theta}$ and $-\pi < \theta \leqslant \pi$.

No curve may cross a branch cut.