Worked Example
Branch Cuts for Multiple Branch Points

What branch cuts would we require for the function

$$f(z) = \log \frac{z-1}{z+1}\ ?$$

It is clear that there are branch points at ± 1, but we have a non-trivial choice of branch cuts. Define $z - 1 = r_1 e^{i\theta_1}$ and $z + 1 = r_2 e^{i\theta_2}$, as shown in the following diagram.

The most straightforward choice is to take two branch cuts, one emanating from each branch point to infinity. In the case shown, we choose $0 \leqslant \theta_1 < 2\pi$ and $-\pi < \theta_2 \leqslant \pi$, and the consequent single-valued definition of $f(z)$ is

$$\begin{aligned} f(z) &= \log(z-1) - \log(z+1) \\ &= (\log r_1 + i\theta_1) - (\log r_2 + i\theta_2) \\ &= \log(r_1/r_2) + i(\theta_1 - \theta_2). \end{aligned}$$

The two cuts make it impossible for z to "wind around" either of the two branch points, so we have obtained a single-valued function which is analytic except along the branch cuts.

The second possible choice is to take only *one* branch cut, between -1 and 1, as shown. This time, we choose both $0 \leqslant \theta_1 < 2\pi$ and $0 \leqslant \theta_2 < 2\pi$ (note that this seems at odds with the location of the branch cut, but this is not a problem as we will explain). The definition of $f(z)$ is as before, but with these different ranges for θ_1 and θ_2.

If z were to cross the branch cut, from above to below say, then θ_1 would be unchanged (at π) but θ_2 would "jump" from 0 to 2π. This is, of course, not allowed, as we may not cross branch cuts. So z cannot wind round just *one* of the branch points.

But it *is* now possible for z to wind around *both* of the branch points together. Consider a curve C which does so. Starting from the point of C on the positive real axis (where $\theta_1 = \theta_2 = 0$) and moving anti-clockwise, both θ_1 and θ_2 increase. When we have made one complete revolution and returned to the positive real axis, having encircled both branch points exactly once, θ_1 and θ_2 both suddenly "jump" from 2π back to 0. But this jump does *not* result in a jump in the value of $\theta_1 - \theta_2$; so $f(z)$ is not affected, and is indeed single-valued as claimed.

Exactly the same choice of branch cuts occurs for the function

$$g(z) = (z^2 - 1)^{1/2}.$$

With the appropriate definitions of θ_1 and θ_2, as above, the single-valued choice is

$$g(z) = (z-1)^{1/2}(z+1)^{1/2} = \sqrt{r_1 r_2}\, e^{i(\theta_1+\theta_2)/2}.$$

This time the single branch cut works because, when both θ_1 and θ_2 jump by 2π, $\frac{1}{2}(\theta_1 + \theta_2)$ jumps by 2π also; and $e^{2\pi i} = 1$. The cut prevents either θ_1 or θ_2 jumping on its own.

This idea can be extended to higher numbers of branch points in the right circumstances.