Summary of Results from Chapter 3: Cartesian Tensors

Transformation Law

If a tensor of rank n has components $T_{ijk\ldots}$ measured in a frame with orthonormal Cartesian axes $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ then its components in a frame with axes $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ are given by

$$T'_{ijk\ldots} = l_{ip} l_{jq} l_{kr} \ldots T_{pqr\ldots}$$

where the rotation matrix $L = (l_{ij})$ is defined by

$$l_{ij} = \mathbf{e}'_i \,.\, \mathbf{e}_j.$$

The components of L satisfy $l_{ik} l_{jk} = \delta_{ij}$ and $l_{ki} l_{kj} = \delta_{ij}$.

The Kronecker Delta and the Alternating Tensor

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is a cyclic permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ is an anticyclic permutation of } (1,2,3) \\ 0 & \text{if any two of } (i,j,k) \text{ are equal} \end{cases}$$

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$[\mathbf{x} \times \mathbf{y}]_i = \epsilon_{ijk} x_j y_k$$

$$\det A = \epsilon_{ijk} a_{1i} a_{2j} a_{3k}$$

Symmetric and Anti-Symmetric Tensors

A tensor $T_{ijk\ldots}$ is symmetric in i, j if $T_{ijk\ldots} = T_{jik\ldots}$ and anti-symmetric in i, j if $T_{ijk\ldots} = -T_{jik\ldots}$.

Any second rank tensor T can be decomposed into a symmetric part S and an anti-symmetric part A where

$$S_{ij} = \tfrac{1}{2}(T_{ij} + T_{ji}),$$

$$A_{ij} = \tfrac{1}{2}(T_{ij} - T_{ji})$$

and

$$T_{ij} = S_{ij} + A_{ij}.$$

Any anti-symmetric second rank tensor A can be expressed in terms of a suitable vector $\boldsymbol{\omega}$ such that $A_{ij} = \epsilon_{ijk}\omega_k$. (In fact, $\omega_k = \frac{1}{2}\epsilon_{klm} A_{lm}$.)

Diagonalisation of Symmetric Second Rank Tensors

If T is a symmetric second rank tensor with eigenvalues (principal values) λ_1, λ_2 and λ_3 and corresponding unit eigenvectors (principal axes) $\mathbf{e}'_1$, $\mathbf{e}'_2$ and $\mathbf{e}'_3$, then the components of T in a frame whose axes coincide with the principal axes are

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Isotropic Tensors

The most general isotropic tensors are:

Rank 0: Any scalar

Rank 1: Only the zero vector

Rank 2: $\lambda\delta_{ij}$

Rank 3: $\lambda\epsilon_{ijk}$

Rank 4: $\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}$

Differential Operators

$\partial_i = \partial/\partial x_i$ is a tensor differential operator of rank one.

$$[\nabla\Phi]_i = \frac{\partial\Phi}{\partial x_i}$$

$$\nabla \cdot \mathbf{F} = \frac{\partial F_i}{\partial x_i}$$

$$[\nabla \times \mathbf{F}]_i = \epsilon_{ijk}\frac{\partial F_k}{\partial x_j}$$

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x_i \partial x_i}$$

$$[\nabla^2\mathbf{F}]_i = \nabla^2(F_i) = \frac{\partial^2 F_i}{\partial x_j \partial x_j}$$