Worked Example
Proving Vector Differential Identities

To prove that $\nabla\cdot(\Phi\mathbf{u}) = \mathbf{u}\cdot\nabla\Phi + \Phi\,\nabla\cdot\mathbf{u}$ where Φ is a scalar field and $\mathbf{u}$ is a vector field:

$$\begin{aligned}
\nabla\cdot(\Phi\mathbf{u}) &= \frac{\partial}{\partial x_i}(\Phi u_i) \\
&= \frac{\partial\Phi}{\partial x_i}u_i + \Phi\frac{\partial u_i}{\partial x_i} \\
&= [\nabla\Phi]_i u_i + \Phi\frac{\partial u_i}{\partial x_i} \\
&= \mathbf{u}\cdot\nabla\Phi + \Phi\,\nabla\cdot\mathbf{u}.
\end{aligned}$$

To prove that $\nabla\times(\mathbf{u}\times\mathbf{v}) = (\nabla\cdot\mathbf{v})\mathbf{u} - (\nabla\cdot\mathbf{u})\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{v}$ where $\mathbf{u}$ and $\mathbf{v}$ are vector fields:

$$\begin{aligned}
[\nabla\times(\mathbf{u}\times\mathbf{v})]_i &= \epsilon_{ijk}\frac{\partial}{\partial x_j}\epsilon_{klm}u_l v_m \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\frac{\partial}{\partial x_j}(u_l v_m) \\
&= \frac{\partial}{\partial x_j}(u_i v_j) - \frac{\partial}{\partial x_j}(u_j v_i) \\
&= u_i\frac{\partial v_j}{\partial x_j} + v_j\frac{\partial u_i}{\partial x_j} - v_i\frac{\partial u_j}{\partial x_j} - u_j\frac{\partial v_i}{\partial x_j} \\
&= [(\nabla\cdot\mathbf{v})\mathbf{u} + (\mathbf{v}\cdot\nabla)\mathbf{u} - (\nabla\cdot\mathbf{u})\mathbf{v} - (\mathbf{u}\cdot\nabla)\mathbf{v}]_i.
\end{aligned}$$