Worked Example
Evaluation of an Isotropic Integral

We wish to calculate

$$T_{ij} = \iiint\limits_{\text{All space}} x_i x_j e^{-r^2} \mathrm{d}V$$

for each value of i and j.

There are no special directions involved either in the domain of integration or in the integrand; so T must be isotropic. Hence $T_{ij} = \lambda\delta_{ij}$ for some λ. To calculate λ, consider $T_{ii} = \lambda\delta_{ii} = 3\lambda$. But we know that

$$\begin{aligned}
T_{ii} &= \iiint\limits_{\mathbb{R}^3} x_i x_i e^{-r^2} \mathrm{d}V \\
&= \iiint\limits_{\mathbb{R}^3} r^2 e^{-r^2} \mathrm{d}V \\
&= \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} r^2 e^{-r^2} r^2 \sin\theta \, \mathrm{d}r \mathrm{d}\theta \mathrm{d}\phi \\
&= 4\pi \int_0^{\infty} r^4 e^{-r^2} \mathrm{d}r \\
&= 4\pi \left(\tfrac{3}{8}\sqrt{\pi}\right).
\end{aligned}$$

Hence we conclude that

$$\iiint\limits_{\mathbb{R}^3} x_i x_j e^{-r^2} \mathrm{d}V = \tfrac{1}{2}\pi\sqrt{\pi}\,\delta_{ij}.$$

Such calculations are often of use when a physical situation has symmetry which can be exploited; for example, consider calculating the inertia tensor of a sphere.