Worked Example
Decomposition of Second Rank Tensors

Consider an elastic body subjected to a simple shear, so that the displacement $\mathbf{u}$ at a location $\mathbf{x} = (x, y, z)$ is given by

$$\mathbf{u} = (\gamma y, 0, 0)$$

for some constant γ. Consider the differential of the displacement, $\partial u_i / \partial x_j$, which is given by the matrix

$$\begin{pmatrix} 0 & \gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We can decompose this tensor into symmetric and anti-symmetric parts,

$$\frac{\partial u_i}{\partial x_j} = \begin{pmatrix} 0 & \frac{1}{2}\gamma & 0 \\ \frac{1}{2}\gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{2}\gamma & 0 \\ -\frac{1}{2}\gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

in which the symmetric part is just the strain tensor e_{ij}. The anti-symmetric part can also be written in the form $\epsilon_{ijk}\omega_k$ where $\boldsymbol{\omega} = (0, 0, \frac{1}{2}\gamma)$.

This decomposition corresponds to writing

$$\mathbf{u} = (\gamma y, 0, 0) = (\tfrac{1}{2}\gamma y, \tfrac{1}{2}\gamma x, 0) + (\tfrac{1}{2}\gamma y, -\tfrac{1}{2}\gamma x, 0).$$

The first term is a stretch at 45° to the (x, y)-axes, while the second is a rotation. In fact, any vector field $\mathbf{u}$ which has zero divergence can be decomposed using this method into a suitable stretch and a solid-body rotation.