Summary of Results from Chapter 2: Poisson's Equation

Physical Origins of Poisson's Equation

Steady-state heat equation	$\nabla^2 T = -S(\mathbf{x})/k$
Steady-state diffusion equation	$\nabla^2 \Phi = -S(\mathbf{x})/k$
Electrostatic potential	$\nabla^2 \Phi = -\rho(\mathbf{x})/\epsilon_0$
Gravitational potential	$\nabla^2 \Phi = 4\pi G\rho(\mathbf{x})$
Flux (in each case)	$-k\nabla\Phi$

Laplace's Equation in 2D Plane Polars

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = 0$$

has general solution

$$\Phi = A_0 + B_0\theta + C_0\ln r + \sum_{n=1}^{\infty}(A_n r^n + C_n r^{-n})\cos n\theta + \sum_{n=1}^{\infty}(B_n r^n + D_n r^{-n})\sin n\theta.$$

Laplace's Equation in 3D Spherical Polars, Axisymmetric Case

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\Phi}{\partial\theta}\right) = 0$$

has general solution

$$\Phi = \sum_{n=0}^{\infty}(A_n r^n + B_n r^{-n-1})P_n(\cos\theta)$$

where P_n are the Legendre polynomials. In particular, $P_0(\cos\theta) = 1$, $P_1(\cos\theta) = \cos\theta$ and $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$.

Uniqueness Theorem for Poisson's Equation

If the problem $\nabla^2\Phi = \sigma(\mathbf{x})$, in a volume V with Dirichlet boundary conditions on the surface S, has a solution for Φ, then that solution is unique.

Green's Function

For a problem with Dirichlet boundary conditions, Green's function $G(\mathbf{x};\mathbf{x}_0)$ satisfies

$$\begin{aligned} \nabla^2 G &= \delta(\mathbf{x} - \mathbf{x}_0) \qquad &\text{in } V, \\ G &= 0 \qquad &\text{on } S. \end{aligned}$$

Fundamental Solutions

Green's function when V is all of space is $\begin{cases} \dfrac{1}{2\pi} \ln |\mathbf{x} - \mathbf{x}_0| + \text{constant} & \text{in 2D,} \\ -\dfrac{1}{4\pi|\mathbf{x} - \mathbf{x}_0|} & \text{in 3D.} \end{cases}$

Images in Circles and Spheres

For Dirichlet boundary conditions, the image point is at

$$\mathbf{x}_1 = \frac{a^2}{|\mathbf{x}_0|^2}\mathbf{x}_0,$$

with strength -1 in 2D and $-a/|\mathbf{x}_0|$ in 3D.

Green's Identity

$$\iiint_V (\Phi\nabla^2\Psi - \Psi\nabla^2\Phi)\,\mathrm{d}V = \iint_S (\Phi\nabla\Psi - \Psi\nabla\Phi)\,\boldsymbol{.}\,\mathbf{n}\,\mathrm{d}S$$

The Integral Solution of Poisson's Equation

The solution to Poisson's equation with Dirichlet boundary conditions,

$$\begin{aligned} \nabla^2\Phi &= \sigma & \text{in } V, \\ \Phi &= f & \text{on } S, \end{aligned}$$

is

$$\Phi(\mathbf{x}_0) = \iiint_V \sigma(\mathbf{x})G(\mathbf{x};\mathbf{x}_0)\,\mathrm{d}V + \iint_S f(\mathbf{x})\frac{\partial G}{\partial n}\,\mathrm{d}S$$

where $G(\mathbf{x};\mathbf{x}_0)$ is Green's function for the problem.

Finite Differences

A first order forward finite difference for $f'(x)$ is

$$\frac{f(x+\delta x) - f(x)}{\delta x}$$

and a second order central finite difference for $f''(x)$ is

$$\frac{f(x+\delta x) - 2f(x) + f(x-\delta x)}{\delta x^2}.$$

Discretization of Poisson's Equation

To solve $\nabla^2\Phi = \sigma(\mathbf{x})$ in a rectangular domain $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$, introduce a grid with $x_i = a + i\delta x$, $y_j = c + j\delta y$. Denote the approximation to $\Phi(x_i, y_j)$ by $\Phi_{i,j}$; if δx and δy are equal then at an interior grid point (i, j) a discretized version of Poisson's equation is

$$\Phi_{i+1,j} + \Phi_{i-1,j} + \Phi_{i,j+1} + \Phi_{i,j-1} - 4\Phi_{i,j} = \sigma(x_i, y_j)\delta x^2.$$