Worked Example
Solution of Laplace's Equation in a 3D Half-Space

We wish to solve $\nabla^2\Phi = 0$ in the half-space $x > 0$ of $\mathbb{R}^3$, with $\Phi = f(y,z)$ on the boundary $x = 0$.

We use the integral solution of Poisson's equation (with $\sigma \equiv 0$) in the half-space, with S being the plane $x = 0$ (strictly speaking, together with the hemisphere at ∞):

$$\Phi(\mathbf{x}_0) = \iiint\limits_V \sigma(\mathbf{x}) G(\mathbf{x};\mathbf{x}_0)\,\mathrm{d}V + \iint\limits_S f(\mathbf{x}) \frac{\partial G}{\partial n}\,\mathrm{d}S$$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y,z)\frac{\partial}{\partial x}G(\mathbf{x};\mathbf{x}_0)\,\mathrm{d}y\,\mathrm{d}z$$

(because $\frac{\partial}{\partial n} = -\frac{\partial}{\partial x}$ on S). To calculate this we need to evaluate

$$\left.\frac{\partial G}{\partial x}\right|_{x=0} = \frac{\partial}{\partial x}\left\{-\frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0|} + \frac{1}{4\pi|\mathbf{x}-\mathbf{x}_1|}\right\}\bigg|_{x=0}$$

$$= \frac{1}{4\pi}\frac{\partial}{\partial x}\left\{-\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} + \frac{1}{\sqrt{(x+x_0)^2+(y-y_0)^2+(z-z_0)^2}}\right\}\bigg|_{x=0}$$

$$= \frac{1}{4\pi}\left\{\frac{x-x_0}{\{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\}^{3/2}} - \frac{x+x_0}{\{(x+x_0)^2+(y-y_0)^2+(z-z_0)^2\}^{3/2}}\right\}\bigg|_{x=0}$$

$$= -\frac{x_0}{2\pi\{x_0^2+(y-y_0)^2+(z-z_0)^2\}^{3/2}}.$$

Therefore

$$\Phi(\mathbf{x}_0) = \frac{x_0}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(y,z)}{\{x_0^2+(y-y_0)^2+(z-z_0)^2\}^{3/2}}\,\mathrm{d}y\,\mathrm{d}z$$

or alternatively (swapping $\mathbf{x}$ and $\mathbf{x}_0$),

$$\Phi(x,y,z) = \frac{x}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(y_0,z_0)}{\{x^2+(y-y_0)^2+(z-z_0)^2\}^{3/2}}\,\mathrm{d}y_0\,\mathrm{d}z_0.$$

This is the solution for:

(i) Steady-state temperature distribution with a wall heated to a specified temperature distribution;

(ii) Steady-state concentration of solute with a wall kept at given concentration;

(iii) Electrostatic potential with a conducting wall held at given potential.