# UNIVERSITY COLLEGE LONDON

*University of London*

# EXAMINATION FOR INTERNAL STUDENTS

*For the following qualifications :-*

*B.Sc.* *M.Sci.*

**Mathematics M234: Electricity and Magnetism**

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| COURSE CODE | : | **MATHM234** |
| UNIT VALUE | : | **0.50** |
| DATE | : | **22-MAY-02** |
| TIME | : | **14.30** |
| TIME ALLOWED | : | **2 hours** |

02-C0949-3-50



**TURN OVER**

*All questions may be attempted but only marks obtained on the best* **four** *solutions will count.*
*The use of an electronic calculator is* **not** *permitted in this examination.*

1. (a) State the Lorentz force law and the vacuum version of Maxwell's equations. Give names for **all** the fields which appear.

   (b) By considering free and bound charges, derive (from the vacuum version) the form of Maxwell's equations in electromagnetic media, taking care to define and name the **D** and **H** fields which occur.

   $$[\textit{Hint: } \rho_{\text{bound}} = -\nabla\cdot\mathbf{P}, \quad \mathbf{j}_{\text{bound}} = \partial\mathbf{P}/\partial t + \nabla\times\mathbf{M}.]$$

   (c) (i) Show that at an interface between two distinct media the boundary condition on the electric displacement is

   $$(\mathbf{D_2} - \mathbf{D_1}) \cdot \hat{\mathbf{n}} \;=\; \sigma,$$

   where $\hat{\mathbf{n}}$ is the unit vector normal to the interface, $\sigma$ is the surface density of free charges, and the subscripts indicate evaluation on either side of the surface.

   (ii) Two media 1 and 2 are separated by a plane boundary. Their permittivities are $\varepsilon_1$ and $\varepsilon_2$, and medium 2 is a conductor. An electric field line in medium 2 makes an angle $\alpha_2$ with the normal to the boundary. Find the angle $\alpha_1$ that the field line makes with the normal in medium 1.

2. (a) State the integral form of Faraday's law of induction. Show how it can be derived from Maxwell's equations. Define all quantities and state the law in words.

   (b) Consider a circular wire loop of radius $r$, threaded by a uniform **B** field perpendicular to the plane of the loop, with magnitude

   $$B(t) = B_0(1 + kt)$$

   where $B_0$ and $k$ are constants. Assume also that the wire loop is being heated in such a way that the radius $r$ is a linear function of time $t$, so $r = vt$, where $v$ is the constant radial velocity of a point on the loop. Determine the induced emf in the wire loop. Indicate on a diagram the directions of $B$ and the induced current.

3. (a) State the Biot-Savart law, which gives the magnetic field $\mathbf{B}$ in terms of an integral of a steady electric current density $\mathbf{j}$. Define the other vectors which appear in the law.

   (b) Show that $\mathbf{B}$ has a vector potential, and therefore that $\nabla\cdot\mathbf{B} = 0$.

   [*Hint:* $\nabla\cdot\nabla\times\mathbf{A} = 0$ for all $\mathbf{A}$.]

   (c) (i) State the integral form of Ampere's law.

   (ii) A cylindrical conductor of radius $a$ and of unit length has a total uniform current $I$ flowing axially. Find the magnetic field $\mathbf{B}$ at points inside the cylinder with radius $r < a$.

4. (a) Define electrostatics. Show that in these circumstances Maxwell's equations can be reduced to a Poisson equation for the electric potential $\phi$.

   (b) Two infinite parallel plates are located perpendicular to the $x$-axis at $x = 0$ and $x = d$ (i.e. they are separated by a distance $d$). The plate at $x = 0$ has potential 0, while the plate at $x = d$ has potential $\phi_0$. The space charge density between the plates is given as

$$\rho = \rho_0 \frac{x}{d},$$

   where the distance $x$ is measured from the plate at zero potential. Find the potential $\phi$ between the plates using Poisson's equation.

5. (a) Starting from the expression for the potential energy of a charge in an electric field, show that the electrostatic energy of a localised charge distribution $\rho$ is given by

$$\mathbf{U_E} = \frac{1}{2}\int_{\text{all space}} \rho(\mathbf{r})\phi(\mathbf{r})dV.$$

   (b) Hence, and using Maxwell's equations in media, show that

$$\mathbf{U_E} = \frac{\varepsilon}{2}\int_{\text{all space}} \|\mathbf{E}\|^2 dV.$$

   (c) Consider a straight wire of conductivity $\sigma$ and radius $r$ oriented along the $z$-axis carrying a steady current $I$. By integrating the Poynting vector over the surface, determine the total power entering a unit length of wire.

   [*Hint:* The magnitude of the magnetic field at the surface is $\frac{\mu_0 I}{2\pi r}$.]

6. (a) State the postulates of the special theory of relativity.

(b) Consider two inertial reference frames, $S$ and $S'$, with $S'$ in motion with velocity $\mathbf{v} = (0, 0, v)$ relative to $S$. Derive the Lorentz transformation law relating the space-time coordinates in the two frames to show that

$$z' = \gamma(z - vt),$$

with a similar equation for $t'$. Define $\gamma$.

(c) A point charge $q$ is at rest at the origin in $S'$. Using the appropriate transformation laws, determine the electric and magnetic fields in $S$ in terms of $\mathbf{r}$.

*Hint:*

$$\mathbf{E}'_{\perp} = \gamma\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)_{\perp} \qquad \mathbf{B}'_{\perp} = \gamma\left(\mathbf{B} - \frac{\mathbf{v} \times \mathbf{E}}{c^2}\right)_{\perp}$$