UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M231: Fluid Mechanics

COURSE CODE	:	**MATHM231**
UNIT VALUE	:	**0.50**
DATE	:	**18-MAY-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

06-C1047-3-130

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count. The fluid is incompressible and inviscid and has constant density ρ. Gravitational acceleration is denoted by g throughout.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Define the terms streamline, particle path and streakline. A fluid moves two-dimensionally so that its velocity $\boldsymbol{u}$ is given by

$$\boldsymbol{u} = (\cosh t)\boldsymbol{i} + (\sinh t)\boldsymbol{j},$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ are the unit vectors for the Cartesian coordinates (x, y).
Obtain equations in terms of x and y alone for the following:

(i) the streamline through (0,1) at time $t = 0$,
(ii) the particle path for a particle released from (0,1) at time $t = 0$,
(iii) the streakline through (0,1) formed by particles released from (0,1) at times $t \leqslant 0$.

Sketch the three loci on the same diagram in the (x, y) plane.

(b) The volume flux F of fluid crossing a closed surface S is defined as

$$F = \int_S \boldsymbol{n} \cdot \boldsymbol{u} \mathrm{d}S,$$

where $\boldsymbol{u}$ is the fluid velocity and $\boldsymbol{n}$ is the normal to the surface S.
By considering an arbitrary sub-region, or otherwise, show that if there are no sources or sinks of fluid within a domain D then $\boldsymbol{u}$ satisfies

$$\nabla \cdot \boldsymbol{u} = 0, \quad \text{within} \quad D.$$

2. (a) Define the circulation about a closed contour within a fluid.

(b) By finding the velocity at large distances and on $|z| = a$, *verify* that the function

$$w = Uz + Ua^2/z - ik \log z$$

gives the complex potential for the flow of a uniform stream of speed U in the positive-x direction past a cylinder of radius a where the circulation about the cylinder is $2\pi k$.
You may find the following relation between the complex velocity potential and the polar components of the velocity useful:

$$e^{i\theta} \frac{dw}{dz} = u_r - iu_\theta.$$

(c) Find the position of the stagnation points in $|z| \geqslant a$ when $k > 0$ and discuss the two cases $2Ua > k > 0$ and $k > 2Ua > 0$.

3. A point vortex of strength κ lies at $z_0 = x_0 + \mathrm{i}y_0$ in the first quadrant i.e. in the quarter plane $0 \leqslant \arg z \leqslant \pi/2$, with solid walls along $\arg z = 0$ and $\arg z = \pi/2$.

(a) Describe the image system for this flow and hence obtain the complex potential for the flow in the first quadrant.

(b) Show that the Cartesian components (u_0, v_0) of the velocity at z_0, *due to the image vortices alone*, are given by

$$u_0 = \kappa x_0^2/[4\pi y_0(x_0^2 + y_0^2)], \qquad v_0 = -\kappa y_0^2/[4\pi x_0(x_0^2 + y_0^2)].$$

(c) The vortex moves as a particle with the velocity derived in (b),

$$\frac{\mathrm{d}x_0}{\mathrm{d}t} = u_0, \qquad \frac{\mathrm{d}y_0}{\mathrm{d}t} = v_0.$$

Find its path and sketch it. You may find it convenient to show first that

$$\frac{\mathrm{d}}{\mathrm{d}t}(x_0^{-2} + y_0^{-2}) = 0.$$

4. Homogeneous incompressible inviscid fluid flows *steadily* under the action of a conservative body force $\mathbf{F}$ with potential G per unit mass (so $\mathbf{F} = -\nabla G$).

(a) Show that

$$\nabla(p/\rho + \frac{1}{2}\boldsymbol{u}^2 + G) = \boldsymbol{u} \wedge \omega,$$

where $\omega = \nabla \wedge \boldsymbol{u}$ is the vorticity.

(b) Deduce that $p/\rho + \frac{1}{2}\boldsymbol{u}^2 + G$ is constant along any streamline.

(c) A long straight pipe of length l has slowly tapering cross-section. It is inclined so that its axis makes an angle α to the horizontal and with its smaller cross-section downwards. The radius of the pipe at its upper end is three times that at its lower end and water is pumped into the upper end at a steady rate through the pipe to emerge at the lower end at atmospheric pressure. If the pumping pressure is three times atmospheric pressure, show that the fluid leaves the pipe with speed U given by

$$U^2 = \frac{81}{40}(gl\sin\alpha + 2p_a/\rho),$$

where p_a denotes atmospheric pressure.

You may use without proof the Euler equations

$$\rho\frac{D\boldsymbol{u}}{Dt} = -\nabla p + \rho\mathbf{F},$$

and the identity

$$\boldsymbol{u}.\nabla\boldsymbol{u} + \boldsymbol{u} \wedge \omega = \nabla(\frac{1}{2}\boldsymbol{u}^2).$$

5. A small-amplitude wave is progressing in the positive x-direction on the surface of water of constant density ρ and infinite depth, so that the equation of the surface is $y = \eta(x,t)$ where y is measured vertically upwards from the undisturbed surface. The linearised dynamic boundary condition at the surface, $y = 0$, can be written

$$\frac{\partial \phi}{\partial t} = -g\eta,$$

where ϕ is the velocity potential and g is the acceleration due to gravity.

(a) State the partial differential equation governing ϕ in the interior of the fluid.

(b) By considering the motion of a particle on the free surface show that the other linearised boundary condition on ϕ at $y = 0$ is

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}.$$

(c) State, and briefly justify, a boundary condition on ϕ as $y \to -\infty$.

(d) If $\eta(x,t) = \epsilon \sin[(2\pi/\lambda)(x - ct)]$ with $\epsilon \ll \lambda$, show that the velocity potential is

$$\phi = -\epsilon c \exp(2\pi y/\lambda) \cos[(2\pi/\lambda)(x - ct)],$$

where the wavespeed, c, is given by the dispersion relation

$$c^2 = \lambda g/2\pi.$$

(e) Suppose that a second wave, given by $\eta_2(x,t) = \epsilon \sin[(2\pi/\lambda)(x + ct)]$, of the same amplitude but propagating in the opposite direction is also present.

(i) *Write down* the velocity potential for the combined motion.

(ii) Show that (to leading order in ϵ) the particle paths are straight lines and sketch these paths for particles at various depths and horizontal positions.

The following identity may be useful:

$$\cos A - \cos B = 2 \sin\left(\frac{1}{2}(A + B)\right) \sin\left(\frac{1}{2}(B - A)\right) .$$

6. Water is flowing horizontally in a channel of uniform width with uniform speed u_1 and depth h_1. The water passes through a hydraulic jump and its speed and depth downstream of the jump are uniform and equal to u_2 and h_2 respectively.

(a) By considering the conservation of mass in a flow volume containing the jump show that

$$u_1 h_1 = u_2 h_2 = Q,$$

for some constant Q.

(b) By balancing the rate of change of momentum in this flow volume with the force acting on the volume, or otherwise, obtain the relation

$$\frac{1}{2}\rho g h_1^2 - \frac{1}{2}\rho g h_2^2 = \rho Q(u_2 - u_1).$$

(c) Hence show that

$$2Q^2/g = h_1 h_2 (h_1 + h_2),$$

(d) Deduce that if $u_1^2 > gh_1$ then $h_2 > h_1$ and $u_2^2 < gh_2$, a transition from super-critical to subcritical flow. You may find it convenient to write

$$\frac{h_2}{h_1}\left(1 + \frac{h_2}{h_1}\right) = \frac{2Q^2}{gh_1^3},$$

and sketch the left side as a function of h_2/h_1, and to consider the equivalent expression for h_1/h_2.