UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M231: Fluid Mechanics

COURSE CODE : **MATHM231**

UNIT VALUE : **0.50**

DATE : **25-MAY-05**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

05-C0965-3-110

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count. The fluid is incompressible and inviscid and, except where noted in Question 4, has constant density ρ. Gravitational acceleration is denoted by g throughout.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Define the terms streamline, particle path and streakline. A fluid moves two-dimensionally so that its velocity $\boldsymbol{u}$ is given by

$$\boldsymbol{u} = \boldsymbol{i} + \pi \cos(t)\boldsymbol{j},$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ are the unit vectors for the Cartesian co-ordinates (x, y).

(i) Obtain equations in terms of x and y alone for the following:

A. the streamline through (0,0) at time $t = 0$,

B. the particle path for a particle released from (0,0) at time $t = 0$,

(ii) Sketch the two loci in (i) on the same diagram in the (x, y) plane.

(iii) Obtain equations in terms of x and y alone for the following:

A. the streakline at $t = 0$ formed by particles released from (0,0) at times $t \leqslant 0$.

B. the streakline at $t = \pi/2$ formed by particles released from (0,0) at times $t \leqslant \pi/2$.

(iv) Sketch the two loci in (iii) on the same diagram in the (x, y) plane.

(b) The velocity field of a fluid is given by

$$\boldsymbol{u} = [2y\exp(y^2)\cos(x^3) + 1]\boldsymbol{i} + [3x^2\sin(x^3)\exp(y^2) - 3]\boldsymbol{j} .$$

(i) Show that this represents a rotational flow of an incompressible fluid.

(ii) Find a velocity potential for the flow or give a reason why none can exist.

(iii) Find a streamfunction for the flow or give a reason why none can exist.

2. An irrotational two-dimensional flow has streamfunction $\psi = Axy$, where A is a constant and x and y are Cartesian coordinates. A solid circular cylinder $r = a$ is introduced, where r is the radial polar coordinate and a is a constant. There is a circulation κ around the cylinder.

(a) Find a streamfunction for the resulting flow.

(b) Show that the tangential velocity on the cylinder is $-2Aa\sin 2\theta + \kappa/(2\pi a)$.

(c) If $\kappa = 2\pi Aa^2$, show that there are 4 stagnation points and find their positions.

3. A line source of strength $2\pi m$ is placed at the point $x = a$ symmetrically between the walls $\arg z = \pm i\pi/3$ which intersect at O. All the fluid from this source passes into a line sink at a small hole in the walls at O.

   (a) Derive the complex potential for the combined flow in the form
   $$w(z) = m \log[(z^3 - a^3)/z^3].$$

   (b) Sketch the streamlines.

   (c) Calculate the fluid velocity at the point $z = a \exp i\pi/3$.

   (d) If the line sink at O is replaced by a line source of the same strength, sketch the streamlines for this second flow and show that there a stagnation point at $z = 2^{-1/3}a$.

4. The Reynolds' transport theorem for a scalar quantity α can be written
   $$\frac{D}{Dt}\int_{\mathcal{V}} \alpha \mathrm{d}V = \int_{\mathcal{V}} \frac{\partial \alpha}{\partial t}\mathrm{d}V + \int_{S} \alpha \boldsymbol{u} \cdot \boldsymbol{n} \mathrm{d}S.$$

   (a) Explain briefly the meaning of each term in this expression.

   (b) By considering the conservation of mass in a fluid of variable density $\rho(x, y, z, t)$, use the Reynolds' transport theorem to derive
   $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0.$$

   (c) Using this result show that the transport theorem for a scalar quantity β can be written
   $$\frac{D}{Dt}\int_{\mathcal{V}} \rho\beta \mathrm{d}V = \int_{\mathcal{V}} \rho \frac{D\beta}{Dt}\mathrm{d}V$$

   (d) By considering the momentum balance in the presence of external forces $\mathbf{F}$ per unit mass, obtain the Euler equations
   $$\rho \frac{D\boldsymbol{u}}{Dt} = -\nabla p + \rho \mathbf{F}.$$

   (e) Using Euler's equations in Cartesian form or otherwise show that under gravitational force alone the free surface of a fluid of constant density in solid body rotation with
   $$\boldsymbol{u} = \Omega(-y\boldsymbol{i} + x\boldsymbol{j}),$$
   is a paraboloid.

You may use without proof the results
$$\int_S p\boldsymbol{n}\mathrm{d}S = \int_{\mathcal{V}} \nabla p \mathrm{d}V$$
and
$$\nabla \cdot (\phi \boldsymbol{u}) = \phi \nabla \cdot \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\phi.$$

5. A small-amplitude wave is progressing in the positive x-direction on the surface of water of density ρ and mean depth h, so that the equation of the surface is $y = \eta(x, t)$ where y is measured vertically upwards from the undisturbed surface. Surface tension of constant value σ causes the pressure in the water at the surface to differ from that in the atmosphere by an amount proportional to the curvature. The pressure at the surface is thus given by

$$p_a - \sigma \frac{\partial^2 \eta}{\partial x^2},$$

where p_a is the constant atmospheric pressure. The linearised dynamic boundary condition at the surface, $y = 0$, can thus be written

$$\frac{\partial \phi}{\partial t} = -g\eta + \frac{\sigma}{\rho} \frac{\partial^2 \eta}{\partial x^2},$$

where ϕ is the velocity potential and g is the acceleration due to gravity.

(a) By considering the motion of a particle on the free surface show that the other linearised boundary condition on ϕ at $y = 0$ is

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}.$$

(b) State, and briefly justify, the boundary condition on ϕ at $y = -h$.

(c) State the partial differential equation satisfied by ϕ.

(d) If $\eta(x, t) = \epsilon \sin[(2\pi/\lambda)(x - ct)]$ with $\epsilon \ll \lambda$, $\epsilon \ll h$, show, *or simply verify*, that the velocity potential is

$$\phi = -\epsilon c \frac{\cosh[(2\pi/\lambda)(y + h)]}{\sinh(2\pi h/\lambda)} \cos[(2\pi/\lambda)(x - ct)],$$

where the wavespeed is given by the dispersion relation

$$\frac{c^2}{gh} = \frac{\lambda}{2\pi h} \left[1 + \frac{\sigma}{\rho g h^2} \left(\frac{2\pi h}{\lambda} \right)^2 \right] \tanh \left(\frac{2\pi h}{\lambda} \right).$$

6. (a) The depth of water in a channel with rectangular cross-section is equal to h at a location where the width is d. If the volume rate of flow is equal to Q, show that

$$\frac{Q^2}{2gd^2} = h^2(H - h),$$

where H is a constant and g is gravity.

(b) Show that, for a range of values of Q, there are two possible values of h, the larger of which lies between $\frac{2}{3}H$ and H.

(c) The channel width increases smoothly downstream until the width is effectively infinite and the fluid is effectively at rest with depth H. Describe the difference in the behaviour of the flow depending on whether h upstream is initially greater or less than $\frac{2}{3}H$.