

**UNIVERSITY COLLEGE LONDON**

University of London

**EXAMINATION FOR INTERNAL STUDENTS**

For The Following Qualifications:–

*M.Sc. PG Dip*

**Mathematics for General Relativity**

**COURSE CODE : MATHG305**

**DATE : 19-MAY-06**

**TIME : 14.30**

**TIME ALLOWED : 2 Hours**

All questions may be attempted but only marks obtained on the best **four** solutions will count.

The use of an electronic calculator is **not** permitted in this examination.

Geodesic Equation ( $U^a = dX^a/d\tau$ )

$$\frac{dU^a}{d\tau} + \Gamma^a_{bc} U^b U^c = 0; \quad (1)$$

$$\frac{dp_a}{d\tau} = \frac{m}{2} (\partial_a g_{bc}) U^b U^c; \quad (2)$$

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}). \quad (3)$$

Geodesics parameterized by  $s$

$$\frac{d^2 X^a}{ds^2} + \Gamma^a_{bc} \frac{dX^b}{ds} \frac{dX^c}{ds} = 0. \quad (4)$$

Schwarzschild metric line element

$$d\tau^2 = \left(1 - \frac{r_s}{r}\right) dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2); \quad r_s = 2GM.$$

Faraday tensor

$$F_{ab} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}; \quad F^{ab} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}.$$

Dual Faraday tensor

$$*F^{ab} \equiv \frac{1}{2} \epsilon^{abcd} F_{cd} = \begin{pmatrix} 0 & +B_x & +B_y & +B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix}.$$

Maxwell Source Equations

$$\partial_b F^{ab} = j_c^a.$$

Internal Maxwell Equations

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0 \quad \text{or} \quad \partial_b *F^{ab} = 0.$$

1. (a) Determine which of the following equations is a valid tensor equation. Describe the errors in the other equations.

(i)  $f = G_a{}^b K_a{}^d H^a{}_c L^a{}_d$

(ii)  $P_a = A_a{}^b B_b + U_c V^d W_d$

(iii)  $X_{ab} = Q^c{}_{bca} + U_b W_a$

(iv)  $h = \partial^a V^a - \partial^b \partial_c Z_b{}^c$

- (b) Describe how the Riemann tensor  $R^a{}_{bcd}$  transforms from unprimed coordinates  $X$  to primed coordinates  $X'$ .

- (c) Consider a two-dimensional manifold  $M$  with coordinates  $X^1$  and  $X^2$ . Suppose the metric is

$$g_{ab} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

Also let  $V^a$  be a vector and  $Q_{ab}, T^{bc}$  be tensors with values

$$V^1 = 4, \quad V^2 = 1;$$

$$Q_{11} = 1, \quad Q_{12} = 3, \quad Q_{21} = 5, \quad S_{22} = 7,$$

$$T^{11} = 0, \quad T^{12} = 2, \quad T^{21} = 4, \quad T^{22} = 6.$$

Find the following:

(i)  $\bar{\mathbf{V}} \cdot \bar{\mathbf{V}}$ ;

(ii)  $R^c{}_a = T^{bc} Q_{ba}$ .

(iii)  $T^a{}_a$ .

- (d) Using index notation prove the vector identity

$$\nabla \cdot (\bar{\mathbf{A}} \times \bar{\mathbf{B}}) = \bar{\mathbf{B}} \cdot \nabla \times \bar{\mathbf{A}} - \bar{\mathbf{A}} \cdot \nabla \times \bar{\mathbf{B}}.$$

- (e) Consider a four-dimensional manifold with an anti-symmetric tensor  $A^{ab}$  and a symmetric tensor  $S^{ab}$ . How many independent components does  $A^{ab}$  have? How many independent components does  $S^{ab}$  have?
- (f) Suppose a tensor  $A$  has components  $A_E^{ab}$  in the Earth frame  $E$  and components  $A_S^{cd}$  in the space frame  $S$ . Show that if  $A_E^{ab}$  is anti-symmetric then  $A_S^{cd}$  will also be antisymmetric.

2. For this question, assume Special Relativity holds (i.e. flat space with  $g_{ab} = \eta_{ab}$ ).
- Write down the Internal Maxwell Equation for  $a = 0, b = 2, c = 3$ , expressing this equation in terms of the electric and magnetic fields.
  - Show that the Internal Maxwell Equations are trivial (i.e., they give no information) if two of the indices are equal (e.g. if  $a = b = 1$ ).
  - Let  $\epsilon^{abcd}$  be the totally antisymmetric Levi-Civita tensor. In terms of this tensor, the dual Faraday tensor is defined as  $*F^{ab} \equiv 1/2 \epsilon^{abcd} F_{cd}$ . Derive a simple expression for  $\partial_b *F^{ab}$  from this definition and the Internal Maxwell Equations, showing your work.
  - Express  $*F^{ab} F_{ab}$  in terms of  $\bar{\mathbf{E}}$  and  $\bar{\mathbf{B}}$  (you may use the expressions for  $F_{ab}$  and  $*F^{ab}$  given on the first page).
  - In terms of the 4-vector potential  $\phi_a$ , the Faraday Tensor  $F_{ab}$  is  $F_{ab} = \partial_b \phi_a - \partial_a \phi_b$ . The helicity four-vector  $h^a$  is defined as

$$h^a = *F^{ab} \phi_b.$$

Show that  $h^a$  is conserved ( $\partial_a h^a = 0$ ) if the electric and magnetic fields are everywhere perpendicular.

3. (a) Consider a particle of mass  $m$  moving in a geodesic around an object of mass  $M$  whose gravitational field is described by the Schwarzschild metric. The particle moves in the plane  $\theta = \pi/2$ . The Schwarzschild metric in this plane has two symmetries. What are they? Show from the geodesic equation or Noether's theorem that the particle's orbit has two conserved quantities  $k$  and  $h$ . Express  $dt/d\tau$  and  $d\phi/d\tau$  in terms of  $k$  and  $h$ .
- (b) Change variables from  $r$  to  $u = 1/r$  and parametrize the orbit by  $\phi$  rather than  $\tau$ . Find the orbit equation for  $(du/d\phi)^2$ .
- (c) Let  $E = mk$  and  $L = mh$ . From the orbit equation, derive the second order equation

$$\frac{d^2 u}{d\phi^2} = \frac{r_s m^2}{2L^2} - u + \frac{3}{2} r_s u^2.$$

- (d) Consider a photon with  $m = 0$ . Show that the photon has a circular orbit at

$$u = u_c = \left( \frac{2}{3r_s} \right).$$

Next suppose that a photon is in a nearby orbit, with  $u = u_c(1 + \epsilon)$  with initial condition  $d\epsilon/d\phi(\phi = 0) = 0$ . What is the differential equation for  $\epsilon$ ? Show that the photon will either spiral in to  $r = 0$  or escape to  $r = \infty$ .

4. Consider a two-dimensional surface embedded in three-dimensional Euclidean space (e.g. the surface of a bowl). Using cylindrical coordinates  $(r, \phi, z)$ , the surface is specified by the function  $z = 2\sqrt{r-1}$  for  $r > 1$ .

- (a) Letting  $X^1 = r$  and  $X^2 = \phi$ , show that the metric of this surface is

$$g_{ab} = \begin{pmatrix} \frac{r}{r-1} & 0 \\ 0 & r^2 \end{pmatrix}.$$

Also find  $g^{bc}$ .

- (b) Calculate the Christoffel symbols  $\Gamma^a_{bc}$  for this metric.
- (c) From the geodesic equation, find the differential equations for a geodesic, i.e. find  $d^2r/ds^2$  and  $d^2\phi/ds^2$  for a geodesic parameterized by  $s$ .
- (d) Show that the metric has a symmetry, and hence there exists a conserved quantity (call it  $K$ ) along each geodesic. Express  $d\phi/ds$  in terms of  $K$ .

5. Consider the surface of the Earth (assumed to be perfectly spherical). In terms of its radius  $R$ , co-latitude  $\theta$  and longitude  $\phi$ , the metric line element is

$$ds^2 = R^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

- (a) Suppose a ship at some position  $(\theta, \phi)$  travels with compass bearing  $\psi$  (e.g.  $\psi = 0$  if the ship is heading North, and  $\psi = \pi/4$  if the ship is heading Northeast). If the ship travels a small distance, with a change  $\delta\theta$  in co-latitude and a change  $\delta\phi$  in longitude, what is the ratio  $\delta\theta/\delta\phi$  in terms of  $\psi$ ?
- (b) A Mercator map projection uses coordinates  $(x, y)$ , where the coordinate transformations are

$$\begin{aligned} x &= a \phi, \\ y &= a \log \cot \frac{\theta}{2} \end{aligned}$$

with  $a$  constant. Find  $\delta y/\delta x$  in terms of  $\delta\theta/\delta\phi$ . Consider a direction on the Mercator map which makes an angle of  $\tilde{\psi}$  with respect to the vertical. Show that  $\tilde{\psi} = \psi$ .

- (c) Find the metric line element  $ds^2$  for the Mercator coordinates  $x$  and  $y$  (hint: you may use the identity  $\sin \theta = 1/\cosh(\log \cot \frac{\theta}{2})$  without proof).
- (d) A *conformally flat* metric in two dimensions has the form

$$g_{ab} = f \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where  $f$  is some function of position. What is the function  $f$  for the Mercator map? An angle  $\eta$  between two vectors  $V^a$  and  $W^b$  can be defined by

$$\cos^2 \eta \equiv \frac{(g_{ab} V^a W^b)^2}{(g_{cd} V^c V^d)(g_{ef} W^e W^f)}.$$

Show that the angle  $\eta$  is independent of the function  $f$ .

- (e) A polar map projection uses coordinates  $(X^1, X^2) = (\rho, \lambda)$  where

$$\rho = R \sin \theta, \lambda = \phi.$$

What is the metric in these coordinates? Is this metric conformally flat?