UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C358: Cosmology

COURSE CODE	:	**MATHC358**
UNIT VALUE	:	**0.50**
DATE	:	**20-MAY-04**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

04-C0929-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

Evolution Equations:

$$\dot{R}^2 + k = \frac{8\pi G}{3}\rho R^2 = \frac{H_0^2}{\rho_{c0}}\rho R^2. \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}R}(\rho R^3) = -3pR^2. \tag{2}$$

$$H(t) \equiv \frac{\dot{R}(t)}{R(t)}; \qquad \rho_c \equiv \frac{3}{8\pi G}H^2; \qquad \Omega(t) \equiv \frac{\rho}{\rho_c}.$$

$$H_0 = h\,100\,\mathrm{km\,s^{-1}\,Mpc^{-1}}.$$

Development angle/horizon coordinate:

$$\xi(t) \equiv \int_0^t \frac{dt'}{R(t')}.$$

Robertson–Walker line element:

$$\mathrm{d}\tau^2 = \mathrm{d}t^2 - R^2(t)\left[\,\mathrm{d}\eta^2 + F^2(\eta)(\,\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right].$$

$$F(\eta) = \begin{cases} \sin\eta & k = +1 \\ \eta & k = 0 \\ \sinh\eta & k = -1 \end{cases}$$

1. (a) Suppose the universe is filled with a quintessence field of density ρ and pressure p, with an equation of state $p = w\rho$, where $w \neq -1$. Ignore all other contributions to the energy density of the universe. Using the evolution equations, find $\rho(R)$ in terms of w. Also find $\rho(z)$, where z is the redshift of objects at time t.

   (b) Assuming that $k = 0$, find $R(t)$ in terms of w.

   (c) What value of w corresponds to cold matter? What is the corresponding $R(t)$?

   (d) What value of w corresponds to isotropic radiation? What is the corresponding $R(t)$?

   (e) Find the present age of the universe t_0 in terms of H_0 and w. Suppose h is observed to be $h = 2/3$. What range of values for w are possible if the universe is determined to be at least 15×10^9 years old?

2. (a) Consider a galaxy of absolute luminosity L emitting light at coordinate r_1, redshift z_1, and time t_1 which we observe at time t_0. Define the Luminosity distance d_L, and show that

$$d_L = R_0 r_1 (1 + z_1).$$

   (b) Calculate $r_1 R_0$ for a flat matter dominated universe ($R(t) = R_0(3H_0 t/2)^{2/3}$). Express your answer in terms of z_1. Hence find d_L in terms of z_1 and H_0.

   (c) If $h = 2/3$, find the approximate luminosity distance of the nearest quasar 3C273 at redshift $z_1 \approx 1/6$.

3. (a) Suppose a sound wave at some wavelength λ_1 starts at $t = 0$ with maximum amplitude, and has its first node at decoupling time $t = t_d$. Assume that the wave has dispersion relation $\omega = c_s\sqrt{k^2 - k_J^2}$ with constant c_s and k_J. Find an expression for λ_1 in terms of t_d and k_J.

   (b) What is a rotation curve for a galaxy? How is it observed? Why do observed rotation curves suggest that some of the matter in galaxies is 'dark'?

   (c) The Planck black-body spectrum is

$$\rho(\nu)d\nu = \frac{8\pi h\nu^3 d\nu}{e^{h\nu/kT} - 1}.$$

   Show that in an expanding universe the spectrum remains black-body. What is $T(R)$?

   (d) Given the Einstein Field Equation

$$R^{ab} - \frac{1}{2}g^{ab}R = 8\pi G T^{ab}, \qquad R = R^a{}_a,$$

   show that

$$R^{ab} = 8\pi G(T^{ab} - \frac{1}{2}g^{ab}T), \qquad T = T^a{}_a.$$

   What is T for an isotropic gas in a locally inertial frame of density ρ and pressure p?

4. The volume of a 3-manifold described by the metric g_{ab} and coordinates x^1, x^2, x^3 is given by

$$\mathcal{V} = \int\int\int \sqrt{|\det g|}dx^1dx^2dx^3 \qquad (3)$$

   (a) What is the volume of a 3-sphere S^3 with radius R?

   (b) Describe how one can measure the radius of a sphere, using only intrinsic measurements on the sphere's surface.

   (c) Consider light emitted at coordinate η_1 from a distant galaxy with cosmological redshift z_1. Show that the relation between η_1 and z_1 is given by

$$\eta_1 = \frac{1}{R_0H_0}\int_0^{z_1}\frac{dz}{E(z)},$$

   where

$$E(z) = \frac{H(z)}{H_0}.$$

   (d) What is R_0H_0 for a $k = +1$ universe?

   (e) Suppose Ω_0 is measured to be precisely $\Omega_0 = 1.02$. Given that for observed cosmic parameters

$$\int_0^\infty \frac{dz}{E(z)} \approx 3.5,$$

   estimate the horizon coordinate $\xi(t_0)$. Using part (a), estimate the fraction of the total volume of the universe which lies inside the horizon.

5. (a) The acceleration parameter Q_0 is defined by

$$Q_0 = \frac{\ddot{R}_0 R_0}{\dot{R}_0^2}.$$

Suppose the universe is matter-dominated, so that $\Omega_0 = \Omega_{m0}$. Express Q_0 in terms of Ω_{m0}.

(b) Next suppose both matter and vacuum energy (with omega parameter $\Omega_{\Lambda 0}$) are important. Show that the first evolution equation can be written

$$\dot{R}^2 + k = \frac{C}{R} + DR^2.$$

What are the constants C and D? Rewrite this equation in the form

$$E = T + V,$$

identifying which terms in the equation correspond to (total energy) E, (kinetic energy) T, and (potential energy) V. Sketch $V(R)$ on a graph, assuming that the maximum value of $V(R)$ is less than $-1/2$. Give a brief explanation for why the expansion of the universe accelerates at later times.

(c) Suppose that $k = 0$ and solve the evolution equation to find $R(t)$ (Hint: consider the variable $S = R^3$.)

END OF PAPER