UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics C358: Cosmology

COURSE CODE	:	**MATHC358**
UNIT VALUE	:	**0.50**
DATE	:	**14-MAY-02**
TIME	:	**10.00**
TIME ALLOWED	:	**2 hours**

02-C0919-3-60

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

Evolution Equations:

$$\dot{R}^2 + k = \frac{8\pi G}{3}\rho R^2 = \frac{H_0^2}{\rho_{c0}}\rho R^2. \tag{1}$$

$$\frac{d}{dR}(\rho R^3) = -3pR^2. \tag{2}$$

$$H(t) \equiv \frac{\dot{R}(t)}{R(t)}; \qquad \rho_c \equiv \frac{3}{8\pi G}H^2; \qquad \Omega(t) \equiv \frac{\rho}{\rho_c}.$$

$$H_0 = h\,100\,\mathrm{km\,s^{-1}\,Mpc^{-1}}.$$

Development angle; horizon coordinate:

$$\xi \equiv \int_0^t \frac{dt'}{R(t')}.$$

Robertson-Walker line element:

$$ds^2 = dt^2 - R^2(t)\left[d\eta^2 + F^2(\eta)(d\theta^2 + \sin^2\theta d\phi^2)\right].$$

$$F(\eta) = \begin{cases} \sin\eta & k > 0; \\ \eta & k = 0; \\ \sinh\eta & k < 0. \end{cases}$$

Radial coordinate r as function of redshift z ($k = \pm 1$) for a matter dominated universe:

$$r = \left(\frac{2\left|\Omega_0 - 1\right|^{1/2}}{\Omega_0^2}\right) \frac{\Omega_0 z + (2 - \Omega_0)\left[1 - \sqrt{1 + z\Omega_0}\right]}{1 + z}.$$

1. (a) Show that for $k = \pm 1$, $R_0 = H_0^{-1}|\Omega_0 - 1|^{-1/2}$.

   (b) Suppose that the universe is matter dominated. What is $\rho(R)$? Solve the evolution equations to find $R(\xi)$ for $k = +1$ in terms of H_0 and Ω_0.

   (c) What is $t(\xi)$ for these solutions?

   (d) What is the maximum radius $R_{\max}$ in terms of H_0 and Ω_0? At what time t_{final} does the universe collapse back to $R = 0$? Suppose that $\Omega_0 = 2$ and $h = 2/3$. What is $R_{\max}$ as measured in light years? What is t_{final} as measured in years?

2. Consider a galaxy of diameter D emitting light at coordinate r_1, redshift z_1, and time t_1 which we observe at time t_0.

   (a) Define the angular diameter distance d_A, and show that in terms of cosmic parameters,
   $$d_A = R_0 r_1 (1+z)^{-1}.$$

   (b) The observed angular diameter δ of the galaxy depends on z_1. Assuming a fixed galactic size D, the angular diameter $\delta(z)$ has a minimum at some redshift $z_{\min}$. What is $z_{\min}$ for a matter-dominated universe with $\Omega_0 = 2$?

   (c) In words, why should we expect that $\theta(z)$ has a minimum?

   (d) Suppose the galaxy at r_1, t_1 has a galactic jet. We observe a blob in the jet move across the sky with an apparent proper motion $\mu = d\delta/dt_0$. If we believe that its true velocity perpendicular to the line of sight is $V_\perp$, what would its distance be in a Euclidean universe? Define proper motion distance d_M. Using the result of part (a), or otherwise, find the proper motion distance to the galaxy in terms of cosmic parameters.

   (e) Suppose $V_\perp$ is fixed, and find μ as a function of redshift z. Show that unlike $\delta(z)$, proper motion $\mu(z)$ has no minimum.

3. (a) The Hubble parameter H_0 is often written $H_0 = h\,100\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ (1 pc = 3.3 light years). Express $100\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ in units of years^{-1}, showing your work. You need only be accurate to within 3% .

(b) If the gravitational potential inside the Milky Way is $\Phi(\vec{x})$, what is the escape velocity? How do observations of the fastest stellar velocities in the Milky Way give clues as to the mass distribution of the galaxy?

(c) Define the rotation curve of a galaxy. How is it observed? What rotation curve would you expect in the outer part of a galaxy if most of the galactic mass were concentrated near the center?

(d) What is meant by a 'standard candle'? List four examples of objects which have been used as standard candles in measuring H_0.

(e) A non-rotating black hole of mass M emits black-body radiation at a temperature $T = \beta M^{-1}$ with β constant. The luminosity of a black-body of surface area A is $L = \sigma A T^4$. If the mass $M = M_0$ at $t = 0$, what is $M(t)$? (You may leave your answer in terms of symbols such as σ and β, rather than numerical values.) What is $L(t)$?

4. Suppose a cluster of N galaxies has masses M_i, $i = 1, \ldots, N$, positions $\vec{x}_i$, and velocities $\vec{v}_i = d\vec{x}_i/dt$. The net kinetic energy is

$$K = \frac{1}{2}\sum_{i=1}^{N} M_i v_i^2,$$

and the net potential energy is

$$W = -\frac{G}{2}\sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N} M_i M_j \frac{1}{r_{ij}}$$

where $\vec{r}_{ij} = \vec{x}_i - \vec{x}_j$.

(a) Show that for a statistically steady distribution

$$2K + W = 0.$$

(b) Now suppose the cluster has a distribution of dark matter, spherically symmetric about $\vec{x} = 0$. The density of the dark matter is assumed to follow the law $\rho_D = CR^{-1}$ where $R = |\vec{x}|$. Redo part (a), with the same definitions for K and W (i.e. K and W only sum over the galaxies, not the dark matter), and show that $2K + W \neq 0$. What is $2K + W$?

(c) Assume that the mass-to-light ratio (M/L) is the same for all galaxies. Also assume that the galaxies are distributed isotropically. We can only observe the line of sight velocity $v_{\ell os}$ rather than the true velocity $\vec{v}$. Express K in terms of (M/L) and the observable quantities L_i and $v_{\ell os}^2$.

The distance between two galaxies r_{ij} can not be directly measured; we can only observe the distance projected on the sky d_{ij}. Show that, on average,

$$\langle r_{ij}^{-1}\rangle = \frac{2}{\pi}\langle d_{ij}^{-1}\rangle.$$

Express W in terms of (M/L), L_i, and d_{ij}^{-1}.

5. (a) Consider light emitted at cosmic time t from a distant galaxy with cosmological redshift z. Show that the relation between t and z is given by

$$t(z) = \frac{1}{H_0}\int_0^z \frac{dz}{(1+z)E(z)},$$

where

$$E(z) = \frac{H(z)}{H_0}.$$

(b) In this problem we will assume that the universe is flat, i.e. $k = 0$. First consider a matter dominated universe, where $\rho = \rho_m$ and $\Omega_0 = \Omega_{m0}$. Show that

$$E(z) = (1+z)^{3/2}.$$

(c) Using the integral expression for $t(z)$, calculate t_0 for a $k = 0$ matter dominated universe.

(d) Find an integral expression for $r(z)$ and hence calculate the horizon coordinate ξ_0.

(e) Next consider a $k = 0$ universe with both matter and vacuum energy, so that $\Omega_0 = 1 = \Omega_{m0} + \Omega_{\Lambda 0}$. What is $E(z)$? Show that at early times, the matter dominates, so that $\rho_m(t) \gg \rho_\Lambda(t)$ for $t \ll t_0$. Show that at later times the vacuum energy dominates , so that $\rho_\Lambda(t) \gg \rho_m(t)$ for $t \gg t_0$.

(f) As an approximation, assume the universe is completely matter dominated up to the present time ($\rho(t) = \rho_m(t)$ for $t < t_0$), but that the universe is completely dominated by vacuum energy in future times ($\rho(t) = \rho_\Lambda(t)$ for $t > t_0$).
What is $R(t)$ for $t > t_0$? What is $d\xi/dt$ for $t > t_0$? Show that $\xi(t)$ asymptotically approaches a limiting value as $t \to \infty$. Find that limiting value.