

**UNIVERSITY COLLEGE LONDON**

University of London

**EXAMINATION FOR INTERNAL STUDENTS**

For The Following Qualifications:-

*B.Sc. M.Sc. M.Sci.*

**Mathematics C348: Mathematics For General Relativity**

**COURSE CODE : MATHC348**

**UNIT VALUE : 0.50**

**DATE : 17-MAY-05**

**TIME : 14.30**

**TIME ALLOWED : 2 Hours**

All questions may be attempted but only marks obtained on the best four solutions will count.

The use of an electronic calculator is **not** permitted in this examination.

Geodesic Equation ( $U^a = dX^a/d\tau$ )

$$\frac{dU^a}{d\tau} + \Gamma^a_{bc} U^b U^c = 0; \quad (1)$$

$$\frac{dp_a}{d\tau} = \frac{m}{2} (\partial_a g_{bc}) U^b U^c; \quad (2)$$

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}). \quad (3)$$

Geodesics parameterized by  $\lambda$

$$\frac{d^2 X^a}{d\lambda^2} + \Gamma^a_{bc} \frac{dX^b}{d\lambda} \frac{dX^c}{d\lambda} = 0. \quad (4)$$

Schwarzschild metric line element

$$d\tau^2 = \left(1 - \frac{r_s}{r}\right) dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2); \quad r_s = 2GM.$$

Faraday tensor

$$F_{ab} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}; \quad F^{ab} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}.$$

Dual Faraday tensor

$$*F^{ab} \equiv \frac{1}{2} \epsilon^{abcd} F_{cd} = \begin{pmatrix} 0 & +B_x & +B_y & +B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix}.$$

Maxwell Source Equations

$$\partial_b F^{ab} = j_c^a.$$

Internal Maxwell Equations

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0 \quad \text{or} \quad \partial_b *F^{ab} = 0.$$

1. (a) Determine which of the following equations is a valid tensor equation. Describe the errors in the other equations.

(i)  $A_{ab} = B_a + C_b + D_{ab}$ ;

(ii)  $M^{ab} = G^{ba} + Q^a R^b$ ;

(iii)  $J_b = T_a^c F_{bc} Z^{ad}$ ;

(iv)  $K^c_b = Y_a Z^{ac} X^a_b$ ;

- (b) Describe how  $V^a W_b$  transforms from unprimed coordinates  $X^a$  to primed coordinates  $X'^a$ . Show that  $V^a W_a$  is a scalar.

- (c) Consider a two-dimensional manifold  $M$  with coordinates  $X^1$  and  $X^2$ . Suppose the tensors  $V^a$ ,  $T_{ab}$ , and  $P^{ab}_c$  have the values

$$V : \quad V^1 = 2; \quad V^2 = 3;$$

$$T : \quad \begin{pmatrix} T_{11} = 2 & T_{12} = 3 \\ T_{21} = 1 & T_{22} = 4 \end{pmatrix};$$

$$P : \quad \begin{pmatrix} P^{11}_1 = 1 & P^{12}_1 = 2 \\ P^{21}_1 = 1 & P^{22}_1 = 0 \end{pmatrix}; \quad \begin{pmatrix} P^{11}_2 = 4 & P^{12}_2 = 0 \\ P^{21}_2 = 1 & P^{22}_2 = 5 \end{pmatrix}$$

Find the following tensors:

(i)  $Q^{bc} = V^b V^c$ ;

(ii)  $R^a_b = T_{bc} Q^{ac}$ ;

(iii)  $S^a = P^{ba}_b$ .

- (d) Suppose  $S^{ab}$  is a symmetric tensor and  $A_{ab}$  is antisymmetric. Derive a simple expression for

$$S^{ab} A_{ab},$$

showing your work.

- (e) Using index notation prove the vector identity

$$\nabla \times \nabla f = 0.$$

2. Suppose a spaceship rest frame  $S$  moves at a velocity  $V$  in the  $x$  direction with respect to the Earth frame  $E$ . Let  $U^a$ ,  $m$ , and  $\vec{p}$  be the 4-velocity, rest mass, and 3-momentum of the spaceship. Also let the 4-acceleration be  $a^a = dU^a/d\tau$ .

- (a) (i) What is  $U_S^a$ ? What is  $U_E^a$ ?  
(ii) Show that  $\vec{U} \cdot \vec{U} = 1$ .  
(iii) Show that  $\vec{U} \cdot \vec{a} = 0$ .  
(iv) Define the energy-momentum form with components  $E$  and  $-\vec{p}$  and show that  $E^2 = |\vec{p}|^2 + m^2$ .
- (b) A metre stick is at rest in the spaceship rest frame  $S$ . Derive an expression for its length as measured in the Earth frame  $E$ . What is its length as measured in  $E$  if  $V = 0.8$ ?
- (c) Suppose the spaceship passes through an electro-magnetic field described by the Faraday tensor  $F_{ab}$  and the dual Faraday tensor  $*F^{ab}$ . Define a four vector

$$B^b = U_a *F^{ab}$$

where  $U_a = g_{ac}U^c$ . Express  $B_E^b$  and  $B_S^b$  in terms of  $\vec{E}$ ,  $\vec{B}$ ,  $V$ , and  $\gamma = (1 - V^2)^{-1/2}$ .

3. (a) Consider a particle of mass  $m_0$  moving in a geodesic around an object of mass  $M$  whose gravitational field is described by the Schwarzschild metric. The particle moves in the plane  $\theta = \pi/2$ . Rewrite the Schwarzschild metric in the coordinates  $(t, x, \phi)$ , where  $x = r/r_s$ .
- (b) Show that along a geodesic there are two conserved quantities  $k$  and  $h$ . Express  $dt/d\tau$  and  $d\phi/d\tau$  in terms of  $k$  and  $h$ .
- (c) Derive an expression for  $dx/d\tau$  in the form

$$\frac{1}{2} \left( \frac{dx}{d\tau} \right)^2 + V(x) = C$$

where  $C$  is a constant. What are the effective potential  $V(x)$  and the effective energy  $C$ ?

- (d) Suppose that  $h = 2r_s$ . Find the radii  $x_1$  and  $x_2$ ,  $x_1 < x_2$  where the effective potential has an extremum (maximum or minimum). Show that the particle can have circular orbits at these radii for particular values of  $C$ . What is the value of  $C$  for a circular orbit at  $x_1$ ?
- (e) Give a rough sketch of  $V(x)$  (again with  $h = 2r_s$ ).
- (f) Show from the sketch, or otherwise, that the outer circular orbit at  $x_2$  is stable. Also show that the inner orbit at  $x_1$  is unstable.

4. The metric line element for Euclidean 3-space  $E^3$  is  $ds^2 = dx^2 + dy^2 + dz^2$ . The two-dimensional surface  $H^2$  embedded in  $E^3$  is parameterized by coordinates  $(X^1, X^2) = (r, \phi)$ , where

$$\begin{aligned}x &= R \cosh r \cos \phi \\y &= R \cosh r \sin \phi \\z &= R \sinh r.\end{aligned}$$

- (a) Express the metric line element in the coordinates  $(r, \phi)$ . What is the metric tensor  $g_{ab}$  in these coordinates?
- (b) Calculate the Christoffel symbols  $\Gamma^a_{bc}$  for this metric.
- (c) From the geodesic equation, find the differential equations for a geodesic, i.e. find  $d^2r/ds^2$  and  $d^2\phi/ds^2$  for a geodesic parameterized by  $s$ .
5. Let  $f$  be a scalar field and  $X^a$  be a vector field. The covariant derivative  $\nabla_b$  acts on these fields as follows:

$$\begin{aligned}\nabla_b f &= \partial_b f \\(\nabla_b X)^a &= \partial_b X^a + \Gamma^a_{bc} X^c.\end{aligned}$$

Also,  $\nabla_b$  obeys the product rule.

- (a) In general relativity, explain what is meant by a locally inertial frame (LIF) near a spacetime event  $P$ ? What two conditions on the metric  $g_{ab}$  apply in an LIF? What values do the Christoffel symbols  $\Gamma^a_{bc}$  take in an LIF?
- (b) Derive the formula for the derivative of a type  $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$  second rank tensor  $M_{cd}$ :

$$(\nabla_b M)_{cd} = \partial_b M_{cd} - \Gamma^a_{bc} M_{ad} - \Gamma^a_{bd} M_{ac}.$$

- (c) Let  $g_{cd}$  be the metric tensor. Calculate  $(\nabla_b g)_{cd}$ , showing your work.