UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C348: Mathematics For General Relativity

COURSE CODE : **MATHC348**

UNIT VALUE : **0.50**

DATE : **19-MAY-04**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

04-C0926-3-50

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

Geodesic Equation ($U^a = \mathrm{d}\mathsf{X}^a/\mathrm{d}\tau$)

$$\frac{\mathrm{d}U^a}{\mathrm{d}\tau} + \Gamma^a{}_{bc} U^b U^c = 0; \tag{1}$$

$$\frac{\mathrm{d}p_a}{\mathrm{d}\tau} = \frac{m}{2}(\partial_a g_{bc})\, U^b U^c; \tag{2}$$

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}). \tag{3}$$

Geodesics parameterized by λ

$$\frac{\mathrm{d}^2\mathsf{X}^a}{\mathrm{d}\lambda^2} + \Gamma^a{}_{bc}\frac{\mathrm{d}\mathsf{X}^b}{\mathrm{d}\lambda}\frac{\mathrm{d}\mathsf{X}^c}{\mathrm{d}\lambda} = 0. \tag{4}$$

Schwarzschild metric line element

$$\mathrm{d}\tau^2 = \left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{r_s}{r}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2); \qquad r_s = 2GM.$$

Faraday tensor

$$F_{ab} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}; \qquad F^{ab} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}.$$

Dual Faraday tensor

$$^*F^{ab} \equiv \frac{1}{2}\epsilon^{abcd}F_{cd} = \begin{pmatrix} 0 & +B_x & +B_y & +B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix}.$$

Maxwell Source Equations

$$\partial_b F^{ab} = j^a_e.$$

Internal Maxwell Equations

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0 \qquad \text{or} \qquad \partial_b {}^*F^{ab} = 0.$$

1. (a) How does a second rank mixed tensor $Q^a{}_b$ transform under a coordinate transformation between unprimed coordinates x^a and primed coordinates x'^a?

(b) The trace of a second rank mixed tensor T is defined by

$$\text{trace}(T) = T^a{}_a.$$

Show using the transformation laws, that trace(T) is a scalar, i.e. that it takes the same value in all coordinate frames.

(c) Given a metric tensor g_{ab}, what is the definition of the tensor g^{cd}? What is the tensor $g_{ab}g^{bc}$?

(d) Suppose the metric tensor g_{ab} has both a symmetric part $g_{Sab} = g_{Sba}$ and an antisymmetric part, $g_{Aab} = -g_{Aba}$,

$$g_{ab} = g_{Sab} + g_{Aab}.$$

Show explicitly that the line element $d\tau^2$ is independent of g_{Aab}.

(e) Consider a two-dimensional manifold M with coordinates x^1 and x^2. Suppose the metric is

$$g_{ab} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

Also let A^a be a vector and $S^a{}_{bc}$ a tensor with values

$$\begin{aligned} A^1 &= -1, \quad A^2 = 2; \\ S^1{}_{11} &= 1, \quad S^1{}_{12} = 0, \qquad S^1{}_{21} = 2, \quad S^1{}_{22} = 3, \\ S^2{}_{11} &= 0, \quad S^2{}_{12} = 1, \qquad S^2{}_{21} = 2, \quad S^2{}_{22} = 2. \end{aligned}$$

Find the following tensors:

(i) A_b;

(ii) $X_c = S^a{}_{ac}$;

(iii) $B^a{}_c = X^a X_b$.

(f) Using index notation prove the vector identity

$$\nabla \cdot (A \times B) = B \cdot \nabla \times A - A \cdot \nabla \times B.$$

(g) Determine which of the following equations is a valid tensor equation. Describe the errors in the other equations.

(i) $A_{ab} = B^{ab} F^c{}_c$;

(ii) $W^a{}_{bc} = D_c E_b F^a$;

(iii) $f = J_b K_a L^a$.

2. (a) What is the definition of the Lorentz group? Show that all Lorentz transformations have determinant ± 1. Give an example of a Lorentz transformation with determinant -1.

   (b) Suppose inertial reference frame B moves at a velocity V in the x direction with respect to inertial frame A.

   Derive the Lorentz transformation between A and B, using the principles of special relativity. You may assume that the transformation is linear, so that

$$\begin{pmatrix} t \\ x \end{pmatrix}_{\mathsf{B}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}_{\mathsf{A}}$$

   for some numbers a, b, c, d. You may also assume that $y_{\mathsf{A}} = y_{\mathsf{B}}$, $z_{\mathsf{A}} = z_{\mathsf{B}}$.

   (c) The Faraday Tensor F_{ab} can be written in terms of the 4-vector potential ϕ_a as $F_{ab} = \partial_b \phi_a - \partial_a \phi_b$. The Maxwell Source Equations are $\partial_b F^{ab} = j^a$ where j^a is the current density.

   (i) Show that $\partial_a j^a = 0$. Describe the physical meaning of this equation.

   (ii) Write the Maxwell Source Equations in terms of ϕ^a instead of F^{ab}. What do the equations reduce to if we assume Lorentz gauge, $\partial_a \phi^a = 0$?

3. The covariant derivative of a covariant second rank tensor M_{cd} is

$$(\nabla_b M)_{cd} = \partial_b M_{cd} - \Gamma^a{}_{bc} M_{ad} - \Gamma^a{}_{bd} M_{ca}.$$

Christoffel symbols are assumed to be symmetric in their last two indices:

$$\Gamma^a{}_{cb} = \Gamma^a{}_{bc}.$$

   (a) Given that the covariant derivative of the metric is always zero, i.e. $(\nabla_b g)_{cd} = 0$ for any choice of b, c, and d, derive the equation

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}).$$

   (b) Consider Euclidean 3-space $\mathbb{E}^3$ with cylindrical coordinates (ρ, ϕ, z). A conical surface within this space can be defined by the equation

$$z = a\rho$$

   with a constant. Let the coordinates on the cone be $(\mathrm{C}^1, \mathrm{C}^2) = (\rho, \phi)$. Show that the metric for the cone in these coordinates is

$$g_{ab} = \begin{pmatrix} 1 + a^2 & 0 \\ 0 & \rho^2 \end{pmatrix}.$$

   (c) Calculate the eight Christoffel symbols $\Gamma^a{}_{bc}$ for this metric.

   (d) Consider a geodesic on this cone, parameterized by arclength s. Find expressions for $\mathrm{d}^2\phi/\mathrm{d}s^2$ and $\mathrm{d}^2\rho/\mathrm{d}s^2$ along the geodesic.

4. The Newtonian orbit equation for a planet at radius r as a function of angle ϕ is

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = \frac{GM}{h^2},$$

where $u = 1/r$ and $h =$ angular momentum/mass. Geodesics in the Schwarzschild metric satisfy the equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = \frac{GM}{h^2} + (3GM)u^2.$$

(a) Consider a planet of mass m moving in a geodesic around the sun (mass M) whose gravitational field is described by the Schwarzschild metric. The planet moves in the ecliptic plane $\theta = \pi/2$. What is the Schwarzschild metric in this plane? Obtain expressions for $\mathrm{d}t/\mathrm{d}\tau$ and $\mathrm{d}\phi/\mathrm{d}\tau$ in terms of two conserved quantities energy/mass k and angular momentum/mass h.

(b) Show that

$$u(\phi) = u_0(1 + \epsilon \sin\phi)$$

is a solution of the Newtonian orbit equation. Find u_0 in terms of r_s and h. At what angle ϕ is perihelion for this solution? What kind of orbit has $\epsilon = 0$?

(c) Let

$$u(\phi) = u_0(1 + \epsilon \sin\phi + y(\phi))$$

where $y(\phi) \ll 1$. Consider an orbit with $\epsilon \ll 1$. Find an approximate differential equation for $y(\phi)$ good to first order in ϵ and y.

(d) What are the complementary functions for the differential equation derived in (c)? What are the particular integrals? Find the solution given conditions at perihelion $y(\pi/2) = 0$, $y'(\pi/2) = 0$.

(e) The solution for $y(\phi)$ includes the precession term

$$(3GM\epsilon u_0)\left(\frac{\pi}{2} - \phi\right)\cos\phi,$$

plus periodic terms. (Hint: these periodic terms can be neglected). Let the next perihelion be at $\phi = 5\pi/2 + \delta\phi$ where $\delta\phi \ll 1$. Give an approximate expression for the precession angle $\delta\phi$.

5. A spaceship accelerates away from Earth in the x direction. The astronauts feel a constant acceleration equal to g, the acceleration at the Earth's surface. For this question, ignore the y and z coordinates. Proper time on the ship is given by τ. The speed of the ship in the Earth's frame is $V(\tau)$. Let the transformation from ship coordinates S to Earth coordinates E be

$$\frac{\partial \mathsf{E}^a}{\partial \mathsf{S}^b} = \begin{pmatrix} \partial t_\mathsf{E}/\partial t_\mathsf{S} & \partial t_\mathsf{E}/\partial x_\mathsf{S} \\ \partial x_\mathsf{E}/\partial t_\mathsf{S} & \partial x_\mathsf{E}/\partial x_\mathsf{S} \end{pmatrix} = \begin{pmatrix} \gamma(\tau) & \gamma(\tau)V(\tau) \\ \gamma(\tau)V(\tau) & \gamma(\tau) \end{pmatrix}$$

where $\gamma(\tau) = (1 - V^2(\tau))^{-1/2}$. The velocity 4-vector in the ship's frame is

$$u^a_\mathsf{S}(\tau) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The covariant form of Newton's second law of motion for an object of mass m subject to a force F^a is

$$m\frac{\mathrm{D}u^a}{\mathrm{D}\tau} = m\left(\frac{\mathrm{d}u^a}{\mathrm{d}\tau} + \Gamma^a{}_{bc}u^b u^c\right) = F^a.$$

(a) The acceleration at the Earth's surface is approximately g $= 10$ m s^{-2}. Derive an approximate expression for this quantity in relativistic units, i.e. in units of year^{-1}.

(b) Let $\phi(\tau)$ be defined by $V(\tau) = \tanh\phi(\tau)$. Write the transformation matrix in terms of $\phi(\tau)$. What is $u^a_E(\tau)$? What is $du^a_E(\tau)/d\tau$?

(c) In the ship's frame, $F^a_S = \begin{pmatrix} 0 \\ mg \end{pmatrix}$. What is $Du^a_S/D\tau$ in the ship's frame? What is $\Gamma^1{}_{00}$?

(d) What is $Du^a_E/D\tau$ in the Earth frame? If we suppose that the Earth frame is a Locally Inertial Frame, what is Γ^a_{Ebc}? Hence find a second expression for $du^a_E/d\tau$ in the Earth's frame.

(e) Using the results above, find the function $\phi(\tau)$. Hence find the function $t_E(\tau)$.