UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C348: Mathematics For General Relativity

COURSE CODE	:	**MATHC348**
UNIT VALUE	:	**0.50**
DATE	:	**23-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0920-3-60

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

Geodesic Equation ($U^a = \mathrm{d}\mathsf{X}^a/\mathrm{d}\tau$):

$$\frac{\mathrm{d}U^a}{\mathrm{d}\tau} + \Gamma^a{}_{bc}\,U^b U^c = 0; \tag{1}$$

$$\frac{\mathrm{d}p_a}{\mathrm{d}\tau} = \frac{m}{2}\left(\partial_a g_{bc}\right) U^b U^c; \tag{2}$$

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad}\left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\right). \tag{3}$$

Geodesics parameterized by λ:

$$\frac{\mathrm{d}^2\mathsf{X}^a}{\mathrm{d}\lambda^2} + \Gamma^a{}_{bc}\frac{\mathrm{d}\mathsf{X}^b}{\mathrm{d}\lambda}\frac{\mathrm{d}\mathsf{X}^c}{\mathrm{d}\lambda} = 0. \tag{4}$$

Schwarzschild metric line element:

$$\mathrm{d}\tau^2 = \left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{r_s}{r}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2); \qquad r_s = 2GM.$$

Faraday tensor:

$$F_{ab} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \qquad F^{ab} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}.$$

Dual Faraday tensor:

$$\mathfrak{F}^{ab} \equiv \frac{1}{2}\epsilon^{abcd}F_{cd} = \begin{pmatrix} 0 & +B_x & +B_y & +B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix}.$$

Maxwell Source Equations:

$$\partial_b F^{ab} = j_e^a.$$

Internal Maxwell Equations:

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0 \qquad \text{or} \qquad \partial_b \mathfrak{F}^{ab} = 0.$$

1. (a) Given a vector $\overline{\mathbf{V}}$ and a gradient ∇f, show that the directional derivative $\overline{\mathbf{V}} \cdot \nabla f$ is a scalar.

(b) Suppose a curve $\gamma(\lambda)$ on a manifold $\mathcal{M}$ has tangent vector $\overline{\mathbf{V}}(\lambda)$. Let the metric on $\mathcal{M}$ be g_{ab} with line element $\mathrm{d}s^2 = g_{ab}\mathrm{dX}^a\mathrm{dX}^b$.

Let f be a scalar field on $\mathcal{M}$.

- Define the norm of $\overline{\mathbf{V}}$, i.e. $|\overline{\mathbf{V}}|$.
- Show that if the curve is parameterized by length, i.e. $\lambda = s$, then $|\overline{\mathbf{V}}|^2 = 1$.
- What is $|\overline{\mathbf{V}}|^2$ if $\lambda \neq s$?

(c) Using index notation prove the vector identity

$$\nabla \cdot (\nabla f \times \nabla g) = 0.$$

(d) Consider a two-dimensional manifold M with coordinates x^1 and x^2. Suppose the tensors $X^a{}_b$, Y^{ab}, and Q_{ab} have the values

$$X^1{}_1 = 2 \quad X^1{}_2 = 2$$
$$X^2{}_1 = 3 \quad X^2{}_2 = 4$$
$$Y^{11} = 0 \quad Y^{12} = 3$$
$$Y^{21} = 1 \quad Y^{22} = 2$$
$$Q_{11} = -1 \quad Q_{12} = 2$$
$$Q_{21} = 7 \quad Q_{22} = 5$$

Also the metric has the values

$$g_{11} = 1, \qquad g_{12} = 2, \qquad g_{22} = -1.$$

Find the following tensors.

(i) $W^{ac} = Y^{bc}X^a{}_b$

(ii) $f = Q_{ab}Y^{ab}$

(iii) X_{ab}

(e) Determine which of the following equations is a valid tensor equation. Describe the errors in the other equations.

(i) $A_{ab} = B_{ba} + g_{ab}D_cD^c$

(ii) $F^c{}_b = G^{ca}H_{da}$

(iii) $f = J_aK_aL^aM^a + N^b{}_b$

(iv) $P_c = \epsilon^{abcd}Q_aR_bS_d$

2. Suppose that magnetic monopoles exist in nature. Then, in addition to the electric charge-current 4-vector $\bar{\mathbf{j}}_e$, there is a magnetic charge-current 4-vector $\bar{\mathbf{j}}_m = (\rho_m, j_{mx}, j_{my}, j_{mz})$ where ρ_m is the magnetic charge density and j_{mx} is the current of magnetic charge in the x direction. The Maxwell equations become

$$\begin{aligned} \partial_b F^{ab} &= j_e^a \\ \partial_b \mathfrak{F}^{ab} &= j_m^a. \end{aligned}$$

(a) Consider the second equation $\partial_b \mathfrak{F}^{ab} = j_m^a$. Find the four equations for $\vec{E}$ and $\vec{B}$ generated by letting $a = 0$, $a = 1$, $a = 2$, and $a = 3$.

(b) Show that magnetic charge is conserved; i.e. show that

$$\partial_a j_m^a = 0.$$

(c) The Lorentz force on a magnetic monopole of charge q_m and 4-velocity U^a is

$$f^a = \frac{\mathrm{d}p^a}{\mathrm{d}\tau} = q_m U_b \mathfrak{F}^{ab}.$$

Find the four equations generated by letting $a = 0$, $a = 1$, $a = 2$, and $a = 3$. Express these in terms of the three-velocity $\vec{V} = \mathrm{d}\vec{x}/dt$ and $\gamma = (1 - V^2)^{-1/2}$.

(d) Show that the Lorentz force in the previous item is perpendicular to $\bar{\mathbf{U}}$ in the sense that

$$\underline{\mathbf{f}} \cdot \bar{\mathbf{U}} = 0.$$

(e) Suppose that the Faraday tensor F_{ab} can be written in the form

$$F_{ab} = \partial_a \phi_b - \partial_b \phi_a$$

for some four-potential $\underline{\phi}$. Show that the magnetic current 4-vector must vanish, i.e. $\bar{\mathbf{j}}_m = 0$.

3. The metric for Euclidean 3-space E^3 in spherical coordinates $(\mathrm{X}^1, \mathrm{X}^2, \mathrm{X}^3) = (r, \theta, \phi)$ is

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}.$$

The Christoffel symbols $\Gamma^1{}_{bc}$ and $\Gamma^2{}_{bc}$ for this metric, which you do not need to calculate, are

$$\Gamma^1{}_{bc} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & -r \sin^2\theta \end{pmatrix}$$

and

$$\Gamma^2{}_{bc} = \begin{pmatrix} 0 & 1/r & 0 \\ 1/r & 0 & 0 \\ 0 & 0 & -\cos\theta \, \sin\theta \end{pmatrix}.$$

(a) Calculate the remaining Christoffel symbols $\Gamma^3{}_{bc}$, showing your work.

(b) Next consider the spherical surface $r = 1$ with coordinates $(x^2, x^3) = (\theta, \phi)$. Using previous results, or otherwise, find the four Christoffel symbols $\tilde{\Gamma}^2{}_{bc}$ and the four symbols $\tilde{\Gamma}^3{}_{bc}$ $(b, c = 2, 3)$ for the spherical surface.

(c) Geodesics on the sphere are great circles. Find $\mathrm{d}^2\theta/\mathrm{d}\lambda^2$ and $\mathrm{d}^2\phi/\mathrm{d}\lambda^2$ for a great circle parameterized by λ.

(d) Show that these geodesics have a conserved quantity.

4. (a) Consider a particle of mass m moving in a geodesic around an object of mass M whose gravitational field is described by the Schwarzschild metric. The particle moves in the plane $\theta = \pi/2$. What is the Schwarzschild metric in this plane? Show that this metric has two symmetries. What are they? Show that there are two conserved quantities k, corresponding to energy/mass, and h, corresponding to angular momentum/mass.

(b) Change variables to $u = 1/r$ and find an expression for $(\mathrm{d}u/\mathrm{d}\phi)^2$.

(c) Suppose $E = km$ and $L = hm$ are the energy and angular momentum of the particle. By taking the limit $m \to 0$ with E and L held fixed, derive the orbit equation for a photon,

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 = \frac{E^2}{L^2} - u^2 + r_s u^3.$$

Solve this equation in the case $r_s = 0$, assuming that $r = 1/u$ reaches its minimum value at $\phi = \pi/2$. Let $b = r_{min}$, the minimum value of r reached by the photon. Express b in terms of E and L.

(d) For $r \gg r_s$ the approximate solution is (you do not need to show this):

$$u = \frac{\sin\phi}{b} + \frac{r_s}{4b^2}\left(3 - 2\sin\phi + \cos 2\phi\right).$$

Suppose light from a distant star passes close to the surface of the sun on its way to a telescope on Earth. Sketch the path of the light, including both the actual and apparent position of the star. What is the net angle Δ through which the light has been deflected?

5. The covariant derivative ∇ obeys the following formulae (you do not need to show these)

$$\begin{aligned}
\nabla_b f &= \partial_b f \\
(\nabla_b \overline{\mathbf{V}})^a &= \partial_b V^a + \Gamma^a{}_{bc} V^c \\
(\nabla_b \underline{\mathbf{W}})_c &= \partial_b W_c - \Gamma^a{}_{bc} W_a \\
(\nabla_b M)_{cd} &= \partial_b M_{cd} - \Gamma^a{}_{bc} M_{ad} - \Gamma^a{}_{bd} M_{ca}
\end{aligned}$$

(a) Show that the covariant derivative of the metric vanishes, i. e. show that $(\nabla_b g)_{cd} = 0$ for any choice of b, c, and d.

(b) Suppose $Q^a{}_c$ is a mixed second rank tensor. Find its covariant derivative

$$(\nabla_b Q)^a{}_c$$

showing your derivation.

(c) Let $\delta^a{}_c$ be the Kronecker-delta (identity) tensor. Show that

$$(\nabla_b \delta)^a{}_c = 0.$$

(d) Using the results above, or otherwise, show that the derivative of the inverse metric tensor g^{ab} also vanishes.