

All questions may be attempted but only marks obtained on the best **four** solutions will count.

The use of an electronic calculator is **not** permitted in this examination.

Geodesic Equation ($u^a = dx^a/d\tau$):

$$\frac{du^a}{d\tau} + \Gamma^a_{bc} u^b u^c = 0; \quad (1)$$

$$\frac{du_a}{d\tau} = \frac{1}{2} (\partial_a g_{bc}) u^b u^c; \quad (2)$$

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}). \quad (3)$$

Schwarzschild metric line element:

$$d\tau^2 = \left(1 - \frac{r_s}{r}\right) dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2); \quad r_s = 2GM.$$

Faraday tensor:

$$F_{ab} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \quad F^{ab} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}.$$

Dual Faraday tensor:

$$\mathfrak{F}^{ab} = \begin{pmatrix} 0 & +B_x & +B_y & +B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix}.$$

Maxwell Source Equations:

$$\partial_b F^{ab} = j^a.$$

Internal Maxwell Equations:

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0.$$

1. (a) Describe how the metric tensor g_{bc} transforms from unprimed coordinates x^a to primed coordinates x'^a .
- (b) Suppose that in some coordinate system the tensor δ^a_b has the form

$$\delta^a_b = \begin{cases} +1 & a = b \\ 0 & a \neq b \end{cases}.$$

Show that it also has this form in any other coordinate system.

- (c) Consider a two-dimensional manifold M with coordinates x^1 and x^2 . Suppose the tensors A^{ab}_c , B_{ab} , and C^a have the values

$$\begin{aligned} A^{11}_1 &= -2, & A^{11}_2 &= 4, & A^{12}_1 &= -1, & A^{12}_2 &= 0, \\ A^{21}_1 &= 3, & A^{21}_2 &= 3, & A^{22}_1 &= 2, & A^{22}_2 &= 5, \\ B_{11} &= 1, & B_{12} &= 0, & B_{21} &= 1, & B_{22} &= -2, \\ C^1 &= 0, & C^2 &= -2. \end{aligned}$$

Also the metric has the values

$$g_{11} = 3, \quad g_{12} = 5, \quad g_{22} = -3.$$

Find the following tensors.

- (i) $P_b = B_{ab}C^a$
- (ii) $f = C^c A^{ab}_c B_{ab}$
- (iii) $Q^a = A^{ab}_b$
- (iv) C_b
- (d) Only one of the following equations is a valid tensor equation. Determine which equation is valid, and describe the errors in the other equations.
- (i) $D_{ab} = T_a W^a_b$
- (ii) $E_{ab} = F^c_{ba} C_c + L^d S_{dba}$
- (iii) $Z_{mn} = Y^a_m{}^a_n$
- (iv) $P_c = K^a_a R_c J^a_a$
- (e) Using index notation prove the vector identity

$$\nabla \times (f \nabla g) = \nabla f \times \nabla g.$$

2. Consider the surface of the Earth (assumed to be perfectly spherical). In terms of its radius R , co-latitude θ and longitude ϕ , the metric line element is

$$ds^2 = R^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

- (a) Suppose a ship at some position (θ, ϕ) travels with compass bearing ψ (here $\psi = 0$ if the ship is heading due North, $\psi = \pi/4$ if the ship is heading Northeast, and so on). If the ship travels a small distance, with a change $\delta\theta$ in co-latitude and a change $\delta\phi$ in longitude, what is the ratio $\delta\theta/\delta\phi$?
- (b) A Mercator map projection uses coordinates (x, y) , where the coordinate transformations are

$$\begin{aligned} x &= \frac{w}{2\pi} \phi, \\ y &= a \log \cot \frac{\theta}{2} \end{aligned}$$

with w, a constants. Find $\delta y/\delta x$ in terms of $\delta\theta/\delta\phi$. Consider a direction on the Mercator map which makes an angle of $\tilde{\psi}$ with respect to the vertical. Show that $\tilde{\psi} = \psi$ only if $a = w/(2\pi)$.

- (c) Find the metric line element ds^2 for the Mercator coordinates x and y with $a = w/(2\pi)$ (hint: you may use the identity $\sin \theta = 1/\cosh(\log \cot \frac{\theta}{2})$ without proof).
- (d) What is the range of the Mercator coordinate x ? What is the range of the Mercator coordinate y ? Briefly explain why Greenland appears to be larger than Africa on a Mercator map, even though in reality it is much smaller.
- (e) A polar map projection uses coordinates $(x^1, x^2) = (\rho, \lambda)$ where

$$\rho = R \sin \theta, \lambda = \phi.$$

What is the metric in these coordinates?

- (f) Find the Christoffel symbols Γ^2_{bc} for the metric in (e), i.e. find

$$\begin{pmatrix} \Gamma^2_{11} & \Gamma^2_{12} \\ \Gamma^2_{21} & \Gamma^2_{22} \end{pmatrix}.$$

3. For this question, assume Special Relativity holds (i.e. flat space with $g_{ab} = \eta_{ab}$).
- Suppose a particle is moving with 3-velocity \vec{V} relative to some inertial frame. What is its 4-velocity u^a ? What is u_b ?
 - Write down the Maxwell Source Equations for $a = 0, 1, 2, 3$, expressing these equations in terms of the electric and magnetic fields and the charge density and current.
 - Show from the Maxwell Source Equation that electric charge is conserved.
 - Consider a charged particle with mass m , 4-momentum (in lowered co-vector form) $p_b = mu_b$, and charge q . The particle travels through an electromagnetic field with Faraday tensor F_{ab} . The Lorentz force on the particle is

$$\frac{dp_b}{d\tau} = qu^a F_{ab}$$

What is $dp_0/d\tau$ in terms of \vec{E} and \vec{B} ? What is $dp_1/d\tau$? Also find dp^0/dt and dp^1/dt .

- Show that the Lorentz force 4-vector is perpendicular to the velocity 4-vector, i.e. show that

$$\frac{dp_b}{d\tau} u^b = 0.$$

- Suppose an inertial reference frame B moves at a velocity $V\hat{x}$ with respect to inertial frame A . Consider the Lorentz transformation from A to B given by

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}_B = \begin{pmatrix} \gamma & -\gamma V & 0 & 0 \\ -\gamma V & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}_A$$

where $\gamma = (1 - V^2)^{-1/2}$.

How do the electric field components E_x , E_y , E_z transform under this Lorentz transformation (show your derivation)?

4. (a) Consider a particle of mass m moving in a geodesic around an object of mass M whose gravitational field is described by the Schwarzschild metric. The particle moves in the plane $\theta = \pi/2$. The Schwarzschild metric in this plane has two symmetries. What are they? Show that the particle's orbit has two conserved quantities h and k .
- (b) Derive an expression for $dr/d\tau$ in the form

$$\frac{1}{2} \left(\frac{dr}{d\tau} \right)^2 + V(r) = C$$

where C is a constant, and the effective potential is

$$V(r) = -\frac{1}{2} \left(\frac{r_s}{r} - \frac{h^2}{r^2} + \frac{r_s h^2}{r^3} \right).$$

What is the effective energy C ?

- (c) Find the radii r_1 and r_2 , $r_1 < r_2$ where the effective potential has an extremum (maximum or minimum). Show that if $C = V(r_1)$ or $C = V(r_2)$ then the requirements for a circular orbit ($dr/d\tau = d^2r/d\tau^2 = 0$) are satisfied. Show that $h \geq \sqrt{3} r_s$ for these orbits. Also show that $r_2 \geq 3r_s$.
- (d) Let $h = 2r_s$. What are r_1 and r_2 ? Show that for the outer orbit at r_2 , $V''(r_2) > 0$ and hence that this orbit is stable. Is the inner orbit at r_1 stable?

5. (a) Consider a spatial manifold. Let $u^a(s) = dx^a/ds$ be the tangent to a curve parameterized by arclength s . The geodesic equation is

$$\frac{du^a}{ds} + \Gamma^a_{bc} u^b u^c = 0,$$

where $d/ds = u^a \partial_a$. From this form of the geodesic equation, derive the second form

$$\frac{du_a}{ds} = \frac{1}{2} (\partial_a g_{bc}) u^b u^c.$$

- (b) Consider a sphere of radius 1 as a 2-dimensional manifold with coordinates $x_1 = \theta$, $x_2 = \phi$, and line-element

$$ds^2 = (d\theta^2 + \sin^2 \theta d\phi^2).$$

Show that geodesics have a conserved quantity (call it H).

- (c) Using the metric line element, or otherwise, derive an equation for $d\theta/ds$ in terms of H .
- (d) Let $X = \cos \theta$. Show that X satisfies

$$\left(\frac{dX}{ds} \right)^2 = 1 - X^2 - H^2.$$

- (e) Obtain a second-order differential equation for $X(s)$ and write down its general solution. Suppose a geodesic starts at co-latitude θ_0 heading due East. Find $X(s)$ and hence $\theta(s)$ explicitly in terms of θ_0 . Show that the total length of the geodesic (i.e. the length needed to go all the way around the sphere once) is independent of θ_0 .