UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C325: Real Fluids

COURSE CODE	:	**MATHC325**
UNIT VALUE	:	**0.50**
DATE	:	**12-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0910-3-40

TURN OVER

All *questions may be attempted but only marks obtained on the best* **four** *solutions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

With the exception of question 4, you may assume that external (body) forces are absent.

1. A container of volume V with fixed solid boundaries is filled with an incompressible viscous fluid with constant density ρ and coefficient of viscosity μ.

   Using the Cauchy equations of motion,

$$\rho \frac{Du_i}{Dt} = \frac{\partial \sigma_{ij}}{\partial x_j},$$

   derive the equation for the rate of kinetic energy dissipation in the form

$$\frac{d}{dt}\int_V \frac{1}{2}\rho u_i u_i dV = -\int_V \sigma_{ij}\frac{\partial u_i}{\partial x_j} dV.$$

   Hence, or otherwise, derive the following equation for the rate of energy dissipation in a Newtonian fluid,

$$\frac{d}{dt}\int_V \frac{1}{2}\rho u_i u_i dV = -\int_V \Phi dV,$$

   where $\Phi = 2\mu e_{ij}e_{ij}$ is the dissipation function and e_{ij} is the rate of strain tensor.

   [Hint: You may assume the constitutive relation,

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij},$$

   in the usual notation.]

   Derive an exact solution of the Navier-Stokes equations for the time-periodic, uni-directional, flow of an incompressible viscous fluid with kinematic viscosity ν past an infinite flat plate oscillating parallel to itself with the speed $u = u_0 \cos(\omega t)$. Here u_0 and ω are constant.

2. Incompressible fluid of kinematic viscosity ν and density ρ is in a channel formed by two parallel plates at $y = 0$ and $y = h$. The fluid is at rest initially. At time $t = 0$ a constant pressure gradient,

$$\frac{\partial p}{\partial x} = -\rho G,$$

is applied and is held constant thereafter. Verify that a uni-directional flow is possible and show that the flow velocity at $t \geq 0$ can be found in the form

$$u(y,t) = U_0(y) + \sum_{n=1}^{\infty} C_n e^{\lambda_n t} \sin\left(\frac{n\pi y}{h}\right).$$

Find the time-independent component, $U_0(y)$, the constants λ_n and the values of the constant coefficients C_1 and C_2 in terms of G, h and ν.

Discuss qualitatively the flow development at early times, $t \to 0+$, with reference to the flow in three separate regions, namely in the main part of the channel and in two thin boundary layers near the channel walls.

3. Incompressible viscous fluid of kinematic viscosity ν is in a two-dimensional steady flow in the (x, y)-plane. The pressure gradient in the flow is absent, $\nabla p \equiv 0$. It is also known that the x-component of the velocity vector is independent of x. Show that the y-component of the velocity vector is then independent of y and derive the following reduced form of the Navier-Stokes equations for the flow:

$$\begin{aligned} v(x)\,u'(y) &= \nu u''(y) \\ u(y)\,v'(x) &= \nu v''(x) \end{aligned}$$

in the usual notation. Hence show that the flow is only possible if either $u(y)$ or $v(x)$ is constant.

Consider the case $u(y) = u_0 = const > 0$. Derive a solution for the v component in the region $-\infty < x \leq 0$ with the conditions $v(-\infty) = v_0 = const > 0, v(0) = 0$. Describe qualitatively the flow that this solution represents.

4. State the assumptions of the lubrication approximation for a steady two-dimensional flow in a thin film of a heavy incompressible viscous fluid on a horizontal flat plate at $y = 0$.

The boundary of the liquid film exposed to air is given by the equation $y = h(x)$. The fluid surface is subject to a constant atmospheric pressure and a prescribed tangential stress, $\mu \partial u/\partial y = \tau(x)$, where μ is the viscosity coefficient of the fluid. Derive, using the lubrication approximation, the equation for the film surface in the form

$$\frac{d}{dx}\left(\frac{\tau}{2}h^2 - \frac{\rho g}{3}h^3\frac{dh}{dx}\right) = 0,$$

where ρ and g are the fluid's density and the acceleration due to gravity, respectively.

The tangential stress is constant, $\tau = \tau_0$, in the region $x \le 1$ where the film thickness is assumed uniform, $h = h_0$. Downstream, in the region $x > 1$, the tangential stress decreases so that $\tau(x) = \tau_0 x^{-1/2}$. Show that the film thickness in the downstream region can be found in the form $h(x) = h_1 x^{1/4}$ and determine the admissible values of the constant coefficient h_1 in this solution in terms of the parameters τ_0, h_0, ρ and g. Show that there are no solutions in this form when

$$\tau_0 < \frac{2}{3}h_0^2\rho g.$$

5. Show that, for slow steady flow of incompressible fluid, the vector potential $\mathbf{A}$ satisfies the equation

$$\nabla^2\left(\nabla^2\mathbf{A}\right) = 0.$$

A circular cylinder of radius r_0 has its centre at the origin of coordinates x, y. The Cartesian velocity components in the flow far from the cylinder are $u = u_0 y, v = 0$, where u_0 is a constant. Show that the streamfunction for the flow around the cylinder can be written as

$$\psi = \frac{u_0}{4}\left[\psi_0(r) + \psi_1(r)\cos(2\theta)\right],$$

in the polar coordinates r, θ. Hence, or otherwise, verify the following solution for the streamfunction:

$$\psi = \frac{u_0}{4}\left[r^2 - r_0^2 - r^{-2}\left(r^2 - r_0^2\right)^2\cos 2\theta - 2r_0^2\ln\left(\frac{r}{r_0}\right)\right].$$

[Hint: You may assume that

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.]$$

END OF PAPER