Orbits

The trajectories of individual stars (sometimes just called orbits) is in general highly chaotic. This can be so even if there is no collective motion at all (f in equilibrium). Actually, it's not difficult to appreciate why. Think about making bread, the baker's dough being a sort of fluid. Dough is incompressible, but that doesn't prevent you stretching it in one direction and shrinking it in others, and then folding it back. So while the dough keeps much the same shape, initially nearby particles within it can be dispersed to widely different parts of it, through the repeated stretching and folding. The same stretching and folding operation can take place in phase space. In fact it appears that phase space is typically riddled with regions where f gets stretched in one directions while being shrunk in others. Thus nearby orbits tend to diverge, and the divergence is exponential in time, which is the technical definition of chaos in dynamical systems. Simulations suggest that the e-folding time of the divergence is comparable to $T_{\rm cross}$, and gets shorter the more particles there are.

In some special situations, there is no chaos, and the system is said to be 'integrable'. If the dynamics is confined to one real-space dimension (hence two phase-space dimensions) then no stretching-and-folding can happen, and orbits are regular. So in a spherical system all orbits are regular. In addition, there are certain potentials (usually referred to as Stäckel potentials) where the dynamics decouples into three effectively one-dimensional systems; so if some equilibrium f generates a Stäckel potential, the orbits will stay chaos-free. Also, small perturbations of non-chaotic systems tend to produce only small regions of chaos,[2] and orbits may be well described through perturbation theory.

In integrable systems there are significant simplifications. Each orbit is (i) confined to a three-dimensional toroidal subspace of six-dimensional phase space, and (ii) fills its torus evenly.[3] Phase space itself is filled by nested orbit-carrying tori—they have to be nested, since orbits can't cross in phase space. Therefore the time-average of each orbit is completely specified once we have specified which torus it is on; this takes three numbers for each orbit, and these are called 'isolating integrals'—they are constants for each orbit of course. Think of the isolating integrals as a coordinate system that parametrizes orbital tori; transformations to a different set of isolating integrals is like a coordinate transformation.

If isolating integrals exist, then any f that depends only on them will automatically satisfy the collisionless Boltzmann equation. Conversely, since orbits fill their tori evenly, any equilibrium f cannot depend on location *on* the tori, it can only depend on the tori themselves, i.e., on the isolating integrals. This result is known as Jeans' theorem.

You should be wary of Jeans' theorem, especially when people tacitly assume it, because as we saw, it assumes that the system is integrable, which is in general not the case.

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[2] If you ever come across the 'KAM theorem', that's basically it.

[3] These two statements are important results from Hamiltonian dynamical systems which we won't try to prove here. But the statements that follow in this section are straightforward consequences of (i) and (ii).

Spherical systems

In spherical systems Jeans' theorem does apply, so f can depend on (at most) three integrals of motion. The simplest case is for f to be a function of energy $E = \frac{1}{2}v^2 + \Phi$ only. (Since we are considering bound systems, $f = 0$ for $E < 0$ always.) To find an equilibrium solution, we only have to satisfy Poisson's equation.

We'll take $G = 1$ for this section, to simplify the expressions a bit. Poisson's equation is now

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = (4\pi)^2 \int_0^{\sqrt{-2\Phi}} f\left(\tfrac{1}{2}v^2 + \Phi\right) v^2\, dv. \tag{2.20a}$$

We can also replace the integral over v by an integral over E:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = (4\pi)^2\sqrt{2}\int_\Phi^0 \sqrt{E-\Phi}\, f(E)\, dE \tag{2.20b}$$

In (2.20a) we take $f(E)$ as given and try to solve for Φ and hence $\rho(r)$; this is a nonlinear differential equation. In (2.20b) we take Φ as given, and try to solve for $f(E)$; this is a linear integral equation.

There are $f(E)$ models in the literature, and you can always concoct a new one by picking some $\rho(r)$, computing $\Phi(r)$ and then solving (2.20b) numerically. Note that the velocity distribution is isotropic for any $f(E)$. If f depends on other integrals of motion, say angular momentum L or its z component, or both—thus $f(E, L^2, L_z)$—then the velocity distribution will be anisotropic, and there are many examples of these around too.

Example [Two spherical isotropic distribution functions] The Plummer model has

$$\Phi(r) = -(r^2 + a^2)^{-\frac{1}{2}}, \qquad \rho(r) = -\frac{3a^2}{4\pi}\Phi^5, \tag{2.21}$$

and the distribution function

$$f(E) = \frac{24\sqrt{2}a^2}{7\pi^3}(-E)^{\frac{7}{2}}. \tag{2.22}$$

can be verified by inserting in (2.20a). Because of the simple functional forms, the Plummer model is occasionally useful for doing rough calculations, but the r^{-5} density profile is much steeper than elliptical galaxies are observed to have.

The isothermal sphere is defined by analogy with a Maxwell-Boltzmann gas, as

$$f(E) = \frac{\rho_0}{(2\pi\sigma^2)^{\frac{3}{2}}}\exp\left(-\frac{E}{\sigma^2}\right) = \frac{\rho_0}{(2\pi\sigma^2)^{\frac{3}{2}}}\exp\left(-\frac{v^2+\Phi}{\sigma^2}\right), \tag{2.23}$$

and σ^2 is like a temperature. Integrating over velocities gives

$$\rho = \rho_0 \exp\left(-\frac{\Phi}{\sigma^2}\right). \tag{2.24}$$

Using this, Poisson's equation becomes

$$\frac{d}{dr}\left(r^2\frac{d\ln\rho}{dr}\right) = -\frac{4\pi}{\sigma^2}r^2\rho, \tag{2.25}$$

for which the solution is

$$\rho(r) = \frac{\sigma^2}{2\pi r^2}, \tag{2.26}$$

or $\sigma^2/(2\pi G r^2)$ if we put back the G. The isothermal sphere has infinite mass! (A side effect of this is that the boundary condition $\Phi(\infty) = 0$ cannot be used, which we why we needed the redundant-looking constant ρ_0 in (2.24) and (2.23).) Nevertheless, it is often used as a model, with some large-r truncation assumed, for the dark halos of disc galaxies. □

The same $\rho(r)$ can be produced by many different f, all having different velocity distributions.

THE JEANS EQUATIONS

Phase space quantities are hard to measure. Much more often we have information only about averages, e.g., bulk velocities and velocity dispersions. So it is useful to derive equations for the quantities

$$\begin{aligned} \rho &= \int f\, d^3\mathbf{v}, \\ \rho\, \langle v_i \rangle &= \int v_i\, f\, d^3\mathbf{v}, \\ \rho\sigma_{ij} &= \int (v_i - \langle v_i \rangle)(v_j - \langle v_j \rangle)\, f\, d^3\mathbf{v}. \end{aligned} \tag{2.27}$$

by taking moments of the collisionless Boltzmann equation (expressed in the cartesian variables x_i and v_i).

Consider first the zeroth moment

$$\int \left(\frac{\partial f}{\partial t} + v_i\, \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}\, \frac{\partial f}{\partial v_i} \right) d^3\mathbf{v}. \tag{2.28}$$

If we integrate the last term by parts (equivalently, apply the divergence theorem) and assume that f and its derivatives vanish for large enough $\mathbf{v}$, the term vanishes. In the middle term we can take the gradient outside the integral. This gives us

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho\, \langle v_i \rangle}{\partial x_i} = 0, \tag{2.29}$$

which is a continuity equation.

Now we consider the first moment

$$\int \left(v_i\, \frac{\partial f}{\partial t} + v_i v_j\, \frac{\partial f}{\partial x_j} - \frac{\partial \Phi}{\partial x_j}\, v_i\, \frac{\partial f}{\partial v_j} \right) d^3\mathbf{v}. \tag{2.30}$$

Again, we integrate the last term by parts, and since

$$\int v_i\, \frac{\partial f}{\partial v_j}\, d^3\mathbf{v} = -\int \delta_{ij} f\, d^3\mathbf{v},$$

we get

$$\frac{\partial \rho\, \langle v_i \rangle}{\partial t} + \frac{\partial \rho\, \langle v_i v_j \rangle}{\partial x_j} = -\rho\, \frac{\partial \Phi}{\partial x_i}\, . \tag{2.31}$$

From this we subtract $\langle v_i \rangle$ times the continuity equation, and then substitute

$$\langle v_i v_j \rangle = \sigma_{ij} + \langle v_i \rangle \langle v_j \rangle\, ,$$

to get

$$\rho\, \frac{\partial \langle v_i \rangle}{\partial t} + \rho\, \langle v_j \rangle\, \frac{\partial \langle v_i \rangle}{\partial x_j} = -\rho\, \frac{\partial \Phi}{\partial x_i} - \frac{\partial \rho\sigma_{ij}}{\partial x_j}\, , \tag{2.32}$$

which is the same as[4]

$$\frac{d\langle \mathbf{v}\rangle}{dt} = -\nabla\Phi - \frac{1}{\rho}\nabla\cdot(\rho\boldsymbol{\sigma}). \tag{2.33}$$

Finally we have an equation that reminds us of ordinary fluid dynamics but also shows us why a stellar fluid is different. An ordinary fluid has

$$\frac{d\langle \mathbf{v}\rangle}{dt} = -\nabla\Phi - \frac{p}{\rho} + \text{viscous terms}. \tag{2.34}$$

where the pressure p arises because of the high rate of molecular encounters, which also leads to the equation of state, and p is isotropic. In a stellar fluid $\nabla\cdot(\rho\boldsymbol{\sigma})$ behaves like a pressure, but it is anisotropic. A related fact is that in the flow of an ordinary fluid the particle paths and streamlines coincide, whereas stellar orbits and the streamlines $\langle \mathbf{v}\rangle$ do not generally coincide.

Example [Useful forms of the hydrodynamic equation] In a steady state axisymmetric system like a disc galaxy we use cylindrical coordinates, and then $\partial/\partial t = \partial/\partial\phi = 0$. Neglecting $\langle v_R\rangle$ and $\langle v_z\rangle$ (which is realistic), we have

$$\begin{aligned} &\frac{1}{R}\frac{\partial}{\partial R}(R\rho\sigma_{RR}) + \frac{\partial}{\partial z}(\rho\sigma_{Rz}) - \frac{\rho}{R}\left(\left\langle v_\phi\right\rangle^2 + \sigma_{\phi\phi}\right) = -\rho\frac{\partial\Phi}{\partial R}, \\ &\frac{\partial}{\partial z}(\rho\sigma_{zz}) + \frac{1}{R}\frac{\partial}{\partial R}(R\rho\sigma_{Rz}) = -\rho\frac{\partial\Phi}{\partial z}. \end{aligned} \tag{2.35}$$

For a steady state spherical system, we use spherical polar coordinates, so $\partial/\partial\theta = \partial/\partial\phi = 0$, and then neglect $\langle v_r\rangle$ and $\langle v_\theta\rangle$. Then we have

$$\frac{d}{dr}(\rho\sigma_{rr}) + \frac{\rho}{r}\left[2\sigma_{rr} - \left(\sigma_{\theta\theta} + \sigma_{\phi\phi} + \left\langle v_\phi\right\rangle^2\right)\right] = -\rho\frac{d\Phi}{dr}. \tag{2.36}$$

These forms are quite useful. Note that Φ is the total potential but $\rho, \langle\mathbf{v}\rangle, \langle\boldsymbol{\sigma}\rangle$ could be for any subpopulation.

As a simple test to see if this apparatus really does work, let us make a crude model of the Milky way halo. We take $\Phi = v_0^2\ln r$, assume $\boldsymbol{\sigma}$ is constant and isotropic with all diagonal components $= \sigma^2$ (say). Then we say $\rho \propto r^{-n}$ and $\left\langle v_\phi\right\rangle = 0$. This gives $\sigma = v_0/\sqrt{n}$. For the Milky Way halo, $\rho \propto r^{-3.5}$, v_0 as measured from gas on circular orbits is 220 km/sec, and rotation is negligible. So we expect $\sigma \simeq 120$ km/sec. And it is. □

Problem 2.3: An E0 galaxy has a total density distribution

$$\rho_{\rm tot}(r) = \frac{\rho_0}{1 + r^2/a^2}.$$

Show that the enclosed mass $M(r) \propto r^3$ for $r \ll a$ and $M(r) \propto r$ for $r \gg a$. [3]

Now take a population of massless test particles in the potential of this galaxy. Assume that this population is spherical, non-rotating, isothermal and isotropic, with velocity dispersion σ in each velocity component. What is the radial density distribution of this test particle population? [8]

At large r the test particle distribution simplifies and its form depends on a dimensionless number. Give a physical interpretation of this number. What is the condition that the density distributions of the test particle population and the galaxy itself have similar forms ar large r? [7]

[4] Note that d/dt is not $\partial/\partial t$, but

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla$$

sometimes called the convective derivative; also sometimes written as D/Dt to emphasize that it's not $\partial/\partial t$.

3. The Interstellar Medium

The interstellar medium (ISM) is a mixture of the primordial gas left over from galaxy formation and the material spewed out by dying stars. It is only a few percent of a galaxy's mass, and very very diffuse ($\lesssim 10^3$ atoms cm^{-3}). But it is very important because it is the stuff that forms stars. It is also the site of varied physical processes that make it observable and fascinate the people studying them.

Gas

Under laboratory conditions, spectral lines with low transition probabilities are 'forbidden' because the excited states get collisionally de-excited before they can radiate. In the ISM, collisional times are typically much longer than the lifetimes of excited states with only forbidden transitions. So forbidden lines are observable from the ISM, and in fact they can dominate the spectrum.

Cold gas emits only in radio, and the most important ISM line of all is the 21 cm line of atomic hydrogen (H I). It comes from the hyperfine split ground state of the hydrogen atom (split because of the coupling of the nuclear and electron spins). The spin flip transition itself cannot be observed in a laboratory, but the split ground state shows up in the hyperfine splitting of the Lyman lines. H I is observed in both emission or in absorption against a background continuum source. One of the uses of H I observations is to measure rotation velocities of gas. Molecular hydrogen (H_2) has no radio lines, which is unfortunate, since it prevents the coldest and densest parts of the ISM being absorbed directly. What saves the situation somewhat is that CO has strong lines at 1.3 mm and 2.6 mm from transition between rotational states, and CO gets used as a tracer of H_2.

Hot gas is readily observed in optical. An important kind of object are H II regions, which partially ionized hydrogen surrounding a very hot young star or stars (O or B). Hot stars produce a large flux of ultraviolet photons, and any Lyman continuum photons(i.e., $\lambda < 912\,\text{Å}$) will photoionize hydrogen. The ionized hydrogen then recombines. But it doesn't have to recombine into the ground state, it can recombine into an excited state and then radiatively decay after that. This process produces a huge variety of observable lines and continuums, of Lyman, Balmer and on through the infrared and into radio. Of each series, the longest wavelength (or α) line will be the strongest, because the transition rate from principal quantum number n is strongest to $n - 1$. Atoms in H II regions can also be collisionally excited. Atomic hydrogen has no levels accessible at collision energies characteristic of H II regions ($T \sim 10^4\,\text{K}$) but N II, O II, S II, O III, Ne III all do. The [O III] lines at 4959Å and 5007Å are particularly prominent.

A planetary nebula is like a compact H II region, except that it surrounds the exposed core of a highly evolved star rather than a hot young star. Because of their bright emission lines and compactness, planetary nebulae can be detected from much greater distances than individual ordinary stars; they are used as sort of tracers of stars.

The photoionization and recombination process in H II regions and planetary nebulae produces, by a happy accident, one Balmer photon for each Lyman continuum photon from the hot star, so the UV flux can be measured by observing an optical spectrum. The reason is basically that the gas is opaque to Lyman photons and transparent to other photons, since almost the H atoms are in the ground state. A Lyman continuum photon initially from the star will get absorbed by a hydrogen atom, producing a free electron. This electron will then be captured into some bound state. If it gets captured to the ground state we are back where we started (with a ground state atom and a Lyman continuum photon), so consider the case where the electron is captured to some $n > 1$ state. Such a capture releases a free-bound continuum photon which then escapes, and leaves an excited state which wants to decay to $n = 1$. If it decays to $n = 1$ bypassing $n = 2$, it will just produce a Lyman photon which will get almost certainly get absorbed again. Only if it decays to some $n > 1$ will a photon escape. In other words, if the decay bypasses $n = 2$ it almost always gets another chance to decay to $n = 2$ and produce a Balmer photon that escapes. The Lyα photons produced by the final decay from $n = 2$ to $n = 1$ random-walk through the gas as they get absorbed and re-emitted again and again. The total Balmer photon flux thus equals the Lyman continuum photon flux. One can then place the source star in an optical-UV colour magnitude diagram, and determine a colour temperature which is called the Zanstra temperature in this context.

H II regions and planetary nebulae also produce thermal continuum radiation. The process that produces this is free-free emission: free electrons in the H II can interact with protons without recombination, and the acceleration of the electrons in this process produces radiation. (Electrons can interact with other electrons in similar fashion as well, but this produces no radiation because the net electric dipole moment doesn't change.) The resulting spectrum is not blackbody because the gas is transparent to free-free photons. In fact the spectrum is quite flat at radio frequencies—this is the same thing as saying that the time scale for free-free encounters is $\ll 1/\nu$ for radio frequency ν.

When an interstellar gas cloud is seen in front of a continuum source, it produces an enormous variety of absorption lines and bands, by no means all of them well understood. Perhaps the most puzzling ones are the so-called diffuse interstellar bands in the infrared; apparently these are similar to what you get if you take bacteria out of the river at Cardiff and stick them in a spectrograph, which led to some interesting speculations some years ago...

EXAMPLE [Cold interstellar CN] Here is a really cute (and slightly poignant) example of what interstellar absorption lines can do for you. Like most heteronuclear molecules, CN has rotational modes which produce radio lines. The radio lines can be observed directly, but more interesting are the optical lines that have been split because of these rotational modes. Observations of cold CN against background stars reveal, through the relative widths of the split optical lines, the relative populations of the rotational modes, and hence the temperature of the CN. The temperature turns out to be 2.73 K, i.e., these cold clouds are in thermal equilibrium with the microwave background. The temperature of interstellar space was first estimated as $\simeq 3$ K in 1941, well before the Big Bang predictions of 1948 and later, but nobody made the connection at the time. □

In the highest density H II regions ($\sim 10^8\,\mathrm{cm}^{-3}$), either very near a young star, or in a planetary-nebula-like system near the evolved star, population inversion between certain states becomes possible. The overpopulated excited state then decays by stimulated emission, i.e., it becomes a maser. An artificial maser or laser uses a cavity with reflecting walls to mimic an enormous system, but in an astrophysical maser the enormous system is available for free; so an astrophysical maser is not directed perpendicular to some mirrors but shines in all directions. But as in an artificial maser, the emission is coherent (hence polarized), with very narrow lines and high intensity. Masers from OH and H_20 are known. Their high intensity and relatively small size makes masers very useful as kinematic tracers.

Finally, we'll just briefly mention synchrotron radiation, which you'll cover in more detail in the high-energy astrophysics part. It's a broad-band non-thermal radiation emitted by electrons gyrating relativistically in a magnetic field, and can be observed in both optical and radio. The photons are emitted in the instantaneous direction of electron motion and polarized perpendicular to the magnetic field. The really spectacular sources of synchrotron emission are systems with jets (young stellar objects with bipolar outflows, or active galactic nuclei). It is synchrotron emission that lights up the great lobes of radio galaxies.

Dust

Interstellar dust consists of particles of silicates or carbon compounds; the largest are $\simeq 0.5\,\mu$m with $\sim 10^4$ atoms, but some appear to have $\lesssim 10^2$ atoms and thus might be thought of as large molecules. Their nastiest property is that they absorb and scatter light, and the observational effect of these two are known as extinction. (Extinction in magnitudes is denoted as A_V for V-band and so on.) Extinction gets less severe for $\lambda \gtrsim 1\,\mu$m as the wavelength gets much longer than the grains, but it is worse for blue than red right. Hence objects are said to be 'reddened' by interstellar extinction. Grains are transparent to X-rays, though. From our location, extinction is worst along the Milky Way disc, and the Galactic Centre is completely opaque to optical observations.

However, extinction by dust does one very useful thing for optical astronomers. Spinning dust grains tend to align with their long axes perpendicular to the local magnetic field. They thus preferentially block light perpendicular to the magnetic field. Thus the observed polarization will tend to be parallel to the magnetic field. Hence polarization measurements of starlight reveal the direction of the magnetic field (or at least the sky-projection of the direction).

Dust also reflects light, with some polarization. This is observable as reflection nebulae, where the stars cannot be seen (at least in optical) but faint diffuse starlight can be seen as reflected by dust.

Light absorbed by dust will be reradiated as a blackbody-ish spectrum. Such a spectrum is observed (from space, by IRAS) as diffuse emission superimposed on a reflected starlight spectrum, but the associated temperature is extremely high—$\sim 10^3\,$K. The interpretation is that some dust grains are so small (< 100 atoms) that a single ultraviolet photon packs enough energy to heat them to $\sim 10^3\,$K, after which these 'stochastically heated' grains cool again by radiating, mostly in the infrared. This process may be part of the explanation for the correlation between infrared and radio continuum luminosities of galaxies (e.g., at 0.1 mm and 6 cm), which seems to be independent of galaxy type. The idea is that ultraviolet photons from the formation of massive stars cause stochastic heating of dust grains, which then reradiate them to give the infrared luminosity. The supernovae resulting from the same stellar populations produce relativistic electrons which produce the radio continuum as synchrotron emission.

Chemical enrichment

The birth and death of stars and what that does to the interstellar medium is a large and very important subject. We won't be able to do it any sort of justice, but just for a sampler let's discuss the effect on chemical evolution of the ISM.

Consider a region of a galaxy, small enough to be fairly homogeneous, but large enough to contain a good sample of stars. Suppose at time t, the total mass of this region is $M_{\rm total}$, $M_{\rm stars}$ in stars and $M_{\rm gas}$ in gas; also say $M_{\rm metal}$ is the part of $M_{\rm gas}$ in metals. Thus the metallicity of the gas is

$$Z \equiv \frac{M_{\rm metal}}{M_{\rm gas}}. \tag{3.1}$$

Now we consider the effect of forming some new stars over some time δt. This time is longer than the time massive stars spend on the main sequence, so the newly formed massive stars are supposed to have already gone supernova and spewed some more metals into the ISM. Let $\delta M_{\rm stars}$ be the change of stellar (or stellar remnant) mass, and let the metal mass contributed to the ISM by this generation of stars be $p\delta M_{\rm stars}$ (p is known as the 'yield' and we will take it to be constant). We want to find the time evolution of Z, from

$$\delta Z = \delta \left(\frac{M_{\rm metal}}{M_{\rm gas}} \right) = \frac{\delta M_{\rm metal} - Z\delta M_{\rm gas}}{M_{\rm gas}} \tag{3.2}$$

We will assume that the system starts with only gas and at $Z = 0$.

The simplest approximation is the 'closed box model', where gas and stars neither enter not leave this region of the galaxy. Then

$$\delta M_{\rm metal} = p\delta M_{\rm stars} - Z\delta M_{\rm stars} = (p - Z)\delta M_{\rm stars} \tag{3.3}$$

and

$$0 = \delta M_{\rm total} = \delta M_{\rm stars} + \delta M_{\rm gas}. \tag{3.4}$$

Inserting these in equation (3.2) gives

$$\delta Z = -p\frac{\delta M_{\rm gas}}{M_{\rm gas}}, \tag{3.5}$$

whence

$$Z = -p \ln \left(\frac{M_{\rm gas}(t)}{M_{\rm gas}(0)} \right). \tag{3.6}$$

In other words,

$$Z = -p \ln (\text{gas fraction}) .$$

Magellanic irregulars fit this reasonably well, and p is estimated to be $\simeq 0.0025$. In spiral galaxies, the gas fraction in the disc increases as we go outwards, and Z is observed to decrease, though perhaps more steeply than this crude model predicts.

The closed box model can also be used to calculate the distribution of stellar metallicities, because the metallicity of each star approximately indicates Z when that star was formed. If we take all the stars now with metallicities less than some Z_1, the sum of their masses equals $M_{\rm stars}(t)$ for the t when Z equalled Z_1. To get $M_{\rm stars}(t)$ we rewrite (3.5) as

$$\delta Z = \frac{p\delta M_{\rm stars}}{M_{\rm gas}(0) - M_{\rm stars}(t)} \tag{3.7}$$

which gives

$$M_{\rm stars}(t) = \left(1 - e^{-Z/p}\right) M_{\rm gas}(0). \tag{3.8}$$

This gives a tolerably good fit for metal-poor globular clusters. But it fails badly for the solar neighbourhood: the most metal rich stars have $Z \simeq Z_\odot \simeq 0.02$, and (3.8) predicts that $\sim 50\%$ of solar neighbourhood stars will have $Z \leq \frac{1}{4} Z_\odot$; in fact only about 2% do. This is known as the 'G-dwarf problem'.

The G-dwarf problem indicates that the closed-box model is an oversimplification, and that loss and/or accretion of material into a star-forming region needs to be considered.

PROBLEM 3.1: In this problem we consider a 'leaky-box' model, which simulates the effect of shocks from supernovae and winds from young massive stars by making gas leave the formerly closed box at a rate proportional to the star formation rate:

$$\delta M_{\rm total} = -c\delta M_{\rm stars}.$$

Use this to work out $M_{\rm gas}(t)$ in terms of $M_{\rm total}(0)$ and $M_{\rm stars}(t)$. Now modify the closed-box relation between $\delta M_{\rm metal}$ and $\delta M_{\rm stars}$ by adding an appropriate leaking term. [6]

Use these two expressions to derive

$$\delta Z = \frac{p\delta M_{\rm stars}}{M_{\rm total}(0) - (1+c)M_{\rm stars}}. \qquad [2]$$

This expression shows that the leaky box model won't solve the G-dwarf problem? Why? [5]

If we allow the box to accrete gas, that does make metal poor stars rarer.

PROBLEM 3.2: In this problem we consider the 'accreting-box' model, another modification of the closed-box model, this time allowing for metal-free gas to be accreted into the system.

From the assumption that no metal enters or leaves the region, relate $\delta M_{\rm metal}$ and $\delta M_{\rm stars}$. Allowing for (metal-free) gas accretion, relate $\delta M_{\rm stars}$ to $\delta M_{\rm total}$ and $\delta M_{\rm gas}$. Use the above to show that

$$\delta Z = \frac{(p - Z)\delta M_{\rm total} - p\delta M_{\rm gas}}{M_{\rm gas}}. \qquad [8]$$

This equation can be solved exactly with some awkwardness, but for us it's enough to consider the simplest case whether the gas accretion rate equals the star formation rate, so $M_{\rm gas}$ stays constant. For this simple case show that Z asymptotes to p. [6]

Can you argue physically why we should expect such behaviour for stellar metallicities in this case? [5]

In fact this model predicts that $\simeq 3\%$ of solar neighbourhood stars will have $Z \leq \frac{1}{4} Z_\odot$.