1. Introducing Galaxies

Galaxies are a slippery topic in astronomy at present. Galaxies are much less well understood than say stars; certainly our understanding is changing (and hopefully improving) noticeably each year. The standard texts/references are *Galactic Astronomy* by Binney and Merrifield, and *Galactic Dynamics* by Binney and Tremaine. *The Physical Universe* by Shu is more elementary, but very insightful and always repays reading.

The reason galaxies are difficult to understand is that they are made of three very different kinds of things. There are stars of course, but there's also the interstellar medium (which produces stars, and is in turn fed by dying stars), and dark matter (about which we know very little, except that it's there). And these three all influence each other. We'll study each of these, and to a small extent how they influence each other. Some galaxies (more of them in earlier epochs) have 'active nucleii' which can vastly outshine the starlight, but we won't go into that—we'll confine ourselves to 'normal' galaxies.

There are three broad categories of galaxies:

Disc galaxies

These have masses of $10^6 M_\odot$ to $10^{12} M_\odot$. The discs brightness tend to be roughly exponential, i.e.,

$$I(R) = I_0 \exp[-R/R_0] \tag{1.1}$$

I_0 is $\sim 10^2 L_\odot\,\mathrm{pc}^{-2}$. The scale radius R_0 is $\simeq 4\,\mathrm{kpc}$ for the Milky Way. The visible component is $\simeq 95\%$ stars (dominated by F and G stars for giant spirals), and the rest dust and gas. The more gas-rich discs have spiral arms, and arms are regions of high gas density that tend to form stars; clumps of nascent stars are observed as H II regions. Disc galaxies have bulges which appear to be much the same as small ellipticals. All disc galaxies seem to be embedded in much larger dark halos; the ratio of total mass to visible stellar mass is $\simeq 5$, but we don't really have a good mass estimate for any disc galaxy.

Elliptical galaxies

These have masses from $10^6 M_\odot$ to $10^{12+} M_\odot$. There are various functional forms around for fitting the surface brightness, of which the best known is the de Vaucouleurs model

$$I(R) = I_0 \exp\left[-(R/R_0)^{\frac{1}{4}}\right]. \tag{1.2}$$

with $I_0 \sim 10^5 L_\odot\,\mathrm{pc}^{-2}$ for giant ellipticals. (To fit to observations, one typically unsquashes the ellipses to circles first. Also, the functional forms are are only fitted to observations over the restricted range in which $I(R)$ is measurable. So don't be surprised to see very different looking functional forms being fit to the same data.) The visible component is almost entirely stars (dominated by K giants for giant ellipticals), but there appears to be dark matter in a proportion similar to disc galaxies. Ellipticals of masses $\lesssim 10^{11} M_\odot$ rotate as fast as you'd expect from their flattening; giant ellipticals rotate much slower, and tend to be triaxial—more on this later.

At the small end of ellipticals, we might put the globular clusters, even though they occur inside galaxies rather than in isolation. These are clusters of masses from $10^4 M_\odot$ to $10^{6.5} M_\odot$, consisting exclusively of very old stars.

Irregulars

Everything else! They tend to have strong emission lines, and their starlight is dominated by B,A and F types. Basically, they look like they've just been shaken up and are responding by forming stars.

Problem 1.1: Instead of functional forms for the surface brightness $I(R)$, people sometimes pick a functional form for the 3D density $\rho(r)$. These are related by the projection

$$I(R) = 2\int_R^\infty \frac{r\rho(r)\,dr}{\sqrt{r^2-R^2}}$$

which is easily worked out numerically if not analytically.

A popular example are the Dehnen models:

$$\rho(r) = \frac{q}{4\pi}\frac{r^q}{r^3(1+r)^{q+1}}$$

where q is an adjustable parameter. Here the normalization is chosen so that ρ integrates to unity. The special case of $q=1$ (called the Jaffe model) is particularly important because it is found to fit the observed $I(R)$ of ellipticals at least as well as de Vaucouleurs' profile.

What is the potential of a mass distribution with a Jaffe $\rho(r)$? [10]

The Dehnen models have an interesting limit as $q \to 0$. What is it? [5]

Example [The fundamental plane for ellipticals] If we assume that all ellipticals have the same constant mass to light ratio and the same form for the mass distribution (only scalable) then $M \propto I_0 R_0^2$, where I_0 is a characteristic surface brightness and R_0 a characteristic radius. The virial theorem implies $M \propto R_0\sigma_0^2$ where σ_0 is a characteristic velocity dispersion (if we assume dispersion dominates rotation). So under these assumptions we'd expect

$$R_0 I_0 \sigma_0^{-2} = \text{constant}. \tag{1.3}$$

Observationally, ellipticals are found to satisfy

$$R_0 I_0^{0.9} \sigma_0^{-1.4} = \text{constant} \tag{1.4}$$

to within observational uncertainties. In the space of $\log R_0, \log I_0, \log\sigma_0$, equation (1.4) is of course a plane, and it is called the fundamental plane. Deviation from the virial prediction presumably has something to do with varying mass to light, but nobody seems to have much idea of why it's a very good correlation in practice.

In diffuse dwarf ellipticals, $I(R)$ falls off faster than in giant ellipticals or compact dwarf ellipticals, so $M \propto I_0 R_0^2$, wouldn't have the same proportionality factor. And observationally, diffuse dwarf ellipticals don't lie on the fundamental plane. □

Problem 1.2: Suppose some category of galaxies has $I(R) = I_0\, f(R/R_0)$ with all galaxies having the same I_0 and function f but different galaxies having different R_0. If the mass to light is constant everywhere then show that

$$L \propto v^4$$

where L is the total luminosity and v is a characteristic velocity. [10]

For spirals, the $L \propto v^4$ relates the total light to the disc rotation velocity (as measured in radio or infrared), and is called the Tully-Fisher relation. In ellipticals (with v identified with the velocity dispersion) it is called the Faber-Jackson relation. Tully-Fisher is important in distance scale work.

Hubble types

On the whole, galaxy classification probably shouldn't be taken as seriously as stellar classification, because there isn't (yet) a clear physical interpretations of what the gradations mean. But some physical properties do clearly correlate with the so-called Hubble types, so it's worth learning about these at least.

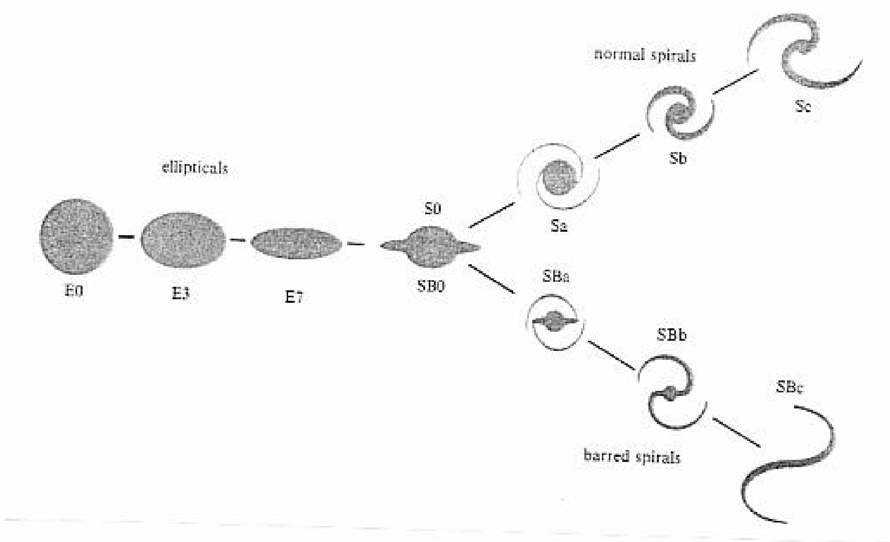

Figure 1.1: The tuning fork diagram of Hubble types.

Figure 1.1 shows the Hubble types. Ellipticals go on the left, labelled as En, where $n = 10(1 - \langle \text{axis ratio} \rangle)$. Then the lenticulars or disc galaxies without spiral arms: S0 and SB0. Then spirals with increasingly spaced arms, Sa etc. if unbarred, SBa etc. if barred.

The left ones are called early types, and the right ones late types. People once thought this represented an evolutionary sequence, but that's long been obsolete. (Our current understanding is that, if anything, galaxies tend to evolve towards early types.) But the old names are still used.

We never see ellipticals flatter than about E7. The reason (as indicated by simulations and normal mode analyses) seems to be that a stellar system any flatter is unstable to buckling, and will eventually settle into something rounder.

Note that bulges get smaller as spiral arms get more widely spaced. Theory for spiral density waves predicts that the spacing between arms is proportional to the disc's mass density.

Handwaving dynamics

Stars are so compact on the scale of a galaxy that a stellar system behaves like a collisionless fluid (except in the cores of galaxies and globular clusters), resembling a plasma in some respects. Gas and dust are collisional. This leads to two very important differences between stellar and gas dynamics in a galaxy.

1) Gas tends to settle into discs, but stars don't.
2) Gravity must be balanced by motion in stellar and gas dynamics, but in equilibrium gas must follow closed orbits (and in the same sense), but stars in general don't. Two streams of stars can go through each other and hardly notice, but two streams of gas will shock (and probably form stars). You could have a disc of stars with no net rotation (just reverse the directions of motion of some stars), but not so with a disc of gas. People sometimes speak of 'rotation-support' and 'pressure-support' balancing self-gravity. Pressure support refers to the high velocity dispersion (compensating for low net-rotation) that comes from reversing stellar motions; this stellar dynamical pressure needn't be isotropic. Observationally gas dispersions are never more than $\simeq 10\,\mathrm{km/sec}$ while stellar dispersions can easily be $\simeq 300\,\mathrm{km/sec}$.

We can start putting together a general picture now. (The rest of this paragraph varies from well-accepted to controversial to wildly speculative, so don't take it too seriously.) Primordial gas will tend to form rotating discs. Differential rotation in the discs will cause spiral density waves, enhancing density along spiral arms and preferentially forming stars. A bulge-less stellar disc is actually unstable to buckling, and produces a bulge with part of its mass. (That's what simulations indicate.) A bulge formed this way will be rotationally supported like the disc that gave rise to it. Meanwhile the disc will continue to form stars, so disc stars will tend to be younger than a bulge stars. Discs that have turned almost all their gas into stars will have stellar discs, but no spiral arms. Now, a disc galaxy can be disrupted by the gravitational influence of another galaxy. It can be a merger of two or more galaxies, or the tidal disruption of a single galaxy; both tending to disrupt discs and produce irregulars with much star formation, then ellipticals. Disruptions of single galaxies will tend to produce rotationally supported ellipticals; but for mergers the angular momentum vectors will tend to cancel, producing pressure support. So we might expect giant ellipticals to be pressure supported. But even a completely gas-free elliptical will generate gas from its dying stars. This second-generation gas will of course settle into discs, and there we might see spiral arms all over again... And all this while, dark matter (whatever it is) will be finding gravitational potential wells in the neighbourhood of galaxies and form halos (sort of like polarization clouds) around them.

Note, by the way, that all galaxies appear to have *some* stars $\sim 10^{10}$yr old. Evidently galaxies all formed fairly early, though they have merged or been otherwise disrupted much more recently.

To end this introductory chapter, let's look at a picture that says rather a lot—it's a very deep photograph of the Sombrero galaxy: Figure 1.2. (You may have across a gorgeous colour poster of this galaxy.) Is it an elliptical with a large embedded disc or a spiral or lenticular with an extra large bulge? But in Figure 1.2 the main galaxy is just an inset within a much larger dark halo. And what is that diffuse fan to the NE and the loop to the SW? Almost certainly traces of past encounters with other galaxies.

Figure 1.2: A recent deep photograph by David Malin of the Sombrero galaxy (aka M 104 and NGC 4594) with a 'normal' image inset to the same scale. The scale bar is $30'$.

2. Stellar Dynamics

A system of stars behaves like a fluid, but one with unusual properties. In a normal fluid two-body interactions are crucial in the dynamics, but stellar encounters are very rare. Instead stellar dynamics is mostly governed by interaction of individual stars with the mean gravitational field of all the other stars.

The virial theorem

Before going into the main material on stellar dynamics, it is worth deriving this basic result. It states for any system of particles bound by an inverse-square force law, the time-averaged kinetic energy (say $\langle T \rangle$) and the time-averaged potential energy (say $\langle V \rangle$) satisfy

$$2\langle T \rangle + \langle V \rangle = 0. \tag{2.1}$$

To prove this, consider the quantity

$$F = \sum_i m_i \dot{\mathbf{x}}_i \cdot \mathbf{x}_i \tag{2.2}$$

where m_i are the masses. Clearly

$$\frac{dF}{dt} = 2T + \sum_i m_i \ddot{\mathbf{x}}_i \cdot \mathbf{x}_i. \tag{2.3}$$

If F is bounded then the long-time average $\langle dF/dt \rangle$ will vanish. Thus

$$2\langle T \rangle + \sum_i m_i \langle \ddot{\mathbf{x}}_i \cdot \mathbf{x}_i \rangle = 0. \tag{2.4}$$

If the system is gravitationally bound, we have

$$2\langle T \rangle - G \sum_{ij} m_i m_j \left\langle \frac{(\mathbf{x}_i - \mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|^3} \cdot \mathbf{x}_i \right\rangle = 0. \tag{2.5}$$

Interchanging the dummy indices in the second term and adding, we have

$$2\langle T \rangle - \tfrac{1}{2} G \sum_{ij} m_i m_j \left\langle \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|} \right\rangle = 0. \tag{2.6}$$

But the second term is now just minus the total potential energy, which proves the result (2.1).

The virial theorem provides an easy way to makes rough estimates of masses, because velocity measurements can give $\langle T \rangle$. But it is prudent to consider virial mass estimates as order-of-magnitude only, because (i) generally one can measure only line-of-sight velocities, and getting $T = \frac{1}{2}\sum_i m_i \dot{\mathbf{x}}_i^2$ from there requires more assumptions (e.g. isotropy of the velocity distribution); and (ii) the systems involved may not be in a steady state, in which case of course the virial theorem does not apply—clusters of galaxies are particularly likely to be quite far from a steady state.

TWO IMPORTANT TIME SCALES

Consider a stellar system of size R, having N stars each of mass m; the stars are distributed roughly homogeneously, with v being a typical velocity, and the system is in dynamical equilibrium. Then from the virial theorem

$$v^2 \simeq NGm/R. \tag{2.7}$$

The crossing time (sometimes called dynamical time)

$$T_{\rm cross} = \frac{R}{v} =\simeq \sqrt{\frac{R^3}{NGm}} \quad \text{or} \quad \frac{1}{\sqrt{G\rho}}. \tag{2.8}$$

The relaxation time is how long it takes for a star's velocity to be changed significantly changed from two-body interactions. To estimate this, consider first one encounter, with a star going past another with impact parameter b. The change δv in the star's velocity due to this encounter is

$$\delta v = Gmb \int_{-\infty}^{\infty} \frac{dt}{(b^2+v^2t^2)^{\frac{3}{2}}} = \frac{2Gm}{bv}. \tag{2.9}$$

(Note that this will be perpendicular to the direction of motion.) Next we consider all the encounters in one crossing time with impact parameters in the range $(b, b+db)$. There are $2Nb\,db/R^2$ of these, since the surface density of stars is $N/(\pi R^2)$. The δv's due these encounters will tend to cancel, so we add their squares and then integrate over b to get the total change in v^2 over one crossing time:

$$\Delta v^2(T_{\rm cross}) = \int_{b_{\rm min}}^{R} \left(\frac{2Gm}{bv}\right)^2 \frac{2N}{R^2} b\,db = 8N\left(\frac{Gm}{Rv}\right)^2 \ln\left(\frac{R}{b_{\rm min}}\right). \tag{2.10}$$

The relaxation time $T_{\rm relax}$ is the time needed for $\Delta v^2 \simeq v^2$. Thus

$$T_{\rm relax} = \frac{v^2}{\Delta v^2(T_{\rm cross})} \times T_{\rm cross} = \frac{1}{8N\ln(R/b_{\rm min})}\frac{(Rv)^3}{(Gm)^2}. \tag{2.11}$$

It's easier to remember $T_{\rm relax}$ in crossing times. Taking $R/b_{\rm min} \simeq N$ and then using equation (2.7) to eliminate R, we get

$$\frac{T_{\rm relax}}{T_{\rm cross}} \simeq \frac{N}{8\ln N}. \tag{2.12}$$

Galaxies are $\lesssim 10^3 T_{\rm cross}$ old and have $\gtrsim 10^6$ stars, so stellar encounters have negligible dynamical effect. In globular clusters, which may have $\sim 10^6$ stars and be $\sim 10^5$ crossing times old, stellar encounters start to become important, and in the cores of globular clusters two-body relaxation is very important.

PROBLEM 2.1: The v and m dependences of the relaxation time can actually be extracted by a back of the envelope calculation.

Consider N stars of mass m each in a box of side R, and let these stars be fixed. Then send another star through this box with speed v. How long does it take for the star to pass near enough to another star that kinetic and two-body potential energies are equal? (Order of magnitude only.) [15]

PROBLEM 2.2: Most researchers doing N-body simulations study the dynamics of galaxies, but some study the dynamics of globular clusters. The latter group of people would seem to have an easier job, because they can easily afford as many particles as there are stars, and they don't have to worry about gas dynamics. So you'd think that globular cluster dynamics would have been cleaned up by now. But in fact, globular cluster dynamics has *not* been cleaned up, and plenty of difficult research remains to be done. This problem is to work out why.

Consider a globular cluster and a galaxy, both $\sim 10^{10}$ yr old. The globular cluster has size ~ 100 pc and $\sim 10^6$ stars with typical velocity $50\,\mathrm{km\,s^{-1}}$. The galaxy has ~ 10 kpc and $\sim 10^{10}$ stars with typical velocity $200\,\mathrm{km\,s^{-1}}$. Now let's say both of these are simulated using 10^6 particles. Can you see two reasons why the globular cluster simulation will be more difficult? [10]

THE COLLISIONLESS BOLTZMANN EQUATION

In the absence of two-body relaxation, stars move under the total gravitational field of all other stars. This field depends only on location in space and we can express it by the potential $\Phi(\mathbf{x})$. Thus the motion of any star is given by Hamilton's equations

$$\frac{d\mathbf{x}}{dt} = \frac{\partial H}{\partial \mathbf{p}}, \qquad \frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{x}}, \tag{2.13}$$

with Hamiltonian

$$H = \frac{p^2}{2m} + \Phi(\mathbf{x}). \tag{2.14}$$

If you haven't met Hamiltonian mechanics before, not to worry: you can easily verify that equations (2.14) and (2.13) give the usual Newtonian equations; but remember the form of equations (2.14).[1] It's very useful to consider the density of stars in 6-dimensional 'phase' space $(\mathbf{x}, \mathbf{p})$; that density is called the distribution function and denoted by f.

Since stars are conserved, f must satisfy a continuity equation:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot \left(f \frac{d\mathbf{x}}{dt} \right) + \frac{\partial}{\partial \mathbf{p}} \cdot \left(f \frac{d\mathbf{p}}{dt} \right) = 0. \tag{2.15}$$

Substituting from Hamilton's equations gives

$$\frac{\partial f}{\partial t} + \frac{d\mathbf{x}}{dt} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{d\mathbf{p}}{dt} \cdot \frac{\partial f}{\partial \mathbf{p}} \equiv \frac{df}{dt} = 0. \tag{2.16}$$

In Hamiltonian dynamics, (2.16) is known as Liouville's theorem, but in stellar dynamics it's usually called the collisionless Boltzmann equation. Physically, it means that if you move with a star, the phase space density around you stays constant. As the sun moves inwards in the Galaxy, the stellar density around it will increase, but at the same time the spread of stellar velocities around it will increase so as to keep phase space density constant.

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[1] Hamiltonian dynamics is a beautiful subject in itself, and helps understand the relations—and differences—between classical mechanics and optics, quantum mechanics, and quantum field theory.

The collisionless Boltzmann equation, and the Poisson equation (which is the gravitational analogue of Gauss's law in electrostatics) together constitute the basic equations of stellar dynamics:

$$\frac{df}{dt} = 0, \qquad \nabla^2\Phi(\mathbf{x}) = 4\pi G\rho(\mathbf{x}). \tag{2.17}$$

EXAMPLE [N-body simulations] You have probably come across N-body simulations of stars in galaxies. The particles in a galaxy simulation do not correspond to stars. They cannot, they have too few particles (10^5 to maybe 10^8 particles max, versus maybe 10^{12} stars in the galaxies being modelled). The appropriate interpretation of simulation particles is as Monte-Carlo samplers of f. Simulation particles have to made collisionless artificially (since there are comparatively few of them, the two-body relaxation time will be correspondingly shorter). The standard way of doing this is to replace the $1/r$ gravitational potential by $(r^2 + a^2)^{-\frac{1}{2}}$, which amounts to smearing out the mass on the 'softening length' scale a.

N-body simulations are widely used now to study the evolution of galaxies, and a trendy research area at present is to incorporate gas dynamics in them. □

Though f is a density in phase space, the full form of the collisionless Boltzmann equation doesn't have to be written in terms of $\mathbf{x}$ and $\mathbf{p}$. We can express df/dt in any set of six variables in phase space.

EXAMPLE [Cylindrical coordinates] In terms of cylindrical coordinates R, ϕ, z and velocities v_R, v_ϕ, v_z we have

$$\frac{\partial f}{\partial t} + \dot{R}\,\frac{\partial f}{\partial R} + \dot{\phi}\,\frac{\partial f}{\partial \phi} + \dot{z}\,\frac{\partial f}{\partial z} + \dot{v}_R\,\frac{\partial f}{\partial v_R} + \dot{v}_\phi\,\frac{\partial f}{\partial v_\phi} + \dot{v}_z\,\frac{\partial f}{\partial v_z} = 0. \tag{2.18}$$

To eliminate the dots we use the standard relations for velocity and acceleration components. We have

$$\begin{aligned} &\dot{R} = v_R, \qquad \dot{\phi} = \frac{v_\phi}{R}, \qquad \dot{z} = v_z \\ &\dot{v}_R = -\,\frac{\partial\Phi}{\partial R} + v_\phi^2, \qquad \dot{v}_\phi = -\frac{1}{R}\,\frac{\partial\Phi}{\partial\phi} - \frac{v_R v_\phi}{R}, \qquad \dot{v}_z = -\,\frac{\partial\Phi}{\partial z}\,, \end{aligned} \tag{2.19}$$

where we have noted substituted $-\nabla\Phi$ for the acceleration. □

You should remember that f is always taken to be a density in six-dimensional phase space, even in situations where it is a function of fewer variables. For example, if f happens to be a function of energy alone, it is not the same as the density in energy space.