King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/3630 General Relativity and Cosmology

Summer 2002

Time Allowed: THREE Hours

Candidates must answer SIX parts of SECTION A, and TWO questions from SECTION B.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

Schwarzschild metric (SM) (in units with $G_N = c = 1$)	$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + +r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$
Energy in Schwarzschild geometry	$\frac{E}{m} = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau}$
SM in Shell coordinates (radial motion):	$ds^2\vert_{\text{radial}} = -dt^2_{\text{shell}} + dr^2_{\text{shell}} \qquad (d\theta = d\phi = 0)$
Christoffel symbols:	$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta})$.
Riemann Curvature Tensor (RCT):	$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} + \Gamma^\alpha{}_{\kappa\mu}\Gamma^\kappa{}_{\beta\nu} - \Gamma^\alpha{}_{\kappa\nu}\Gamma^\kappa{}_{\beta\mu}$.
Properties of RCT:	$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}$.
Ricci tensor:	$R_{\mu\nu} = R_{\nu\mu} = R^\alpha{}_{\mu\alpha\nu}$.
Cosmic Horizon in Friedmann-Robertson-Walker Universe:	$\delta(t) = a(t) \int_{t_0}^{\infty} \frac{dt'}{a(t')}$.

SECTION A - Answer SIX parts of this section

1.1) Consider the two-dimensional metric space $ds^2 = dr^2 - r^2 du^2$, where (r, u) are some coordinates. Without calculating the Christoffel symbols, but by invoking the coordinate transformation $(r, u) \to (x, t)$, where $x = r\cosh u$, $t = r\sinh u$, show that the geodesics of this spacetime are straight lines.

[7 marks]

1.2) Explain briefly why there are no gravitational waves emitted from a spherically symmetric pulsating star.

[7 marks]

1.3) Explain qualitatively what happens to the wavelength of a photon emitted radially at time t_1 and received at time t_2 by observers at rest with respect to an expanding (flat) Robertson–Walker spacetime described by the metric

$$ds^2 = -dt^2 + a^2(t)\left\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right\}.$$

[7 marks]

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1.4) In d spacetime dimensions, the *vacuum* Einstein Equations read

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 0.$$

Show that

$$(d-2)R = 0.$$

What do you conclude about R and $R_{\mu\nu}$ from this analysis, in $d = 2$ spacetime dimensions and in $d > 2$ spacetime dimensions?

[7 marks]

1.5) Assume that, under general coordinate transformations $x^\mu \to x'^\mu(x^\nu)$, a covector $\mathcal{V}_\mu$ and a covariant second rank tensor $\mathcal{T}_{\mu\nu}$ transform as follows: $\mathcal{V}_\mu \to \mathcal{V}\,'_\mu = \frac{\partial x^\alpha}{\partial x'^\mu} \mathcal{V}_\alpha$, and $\mathcal{T}_{\mu\nu} \to \mathcal{T}\,'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \mathcal{T}_{\alpha\beta}$, respectively. Show that the object $\partial_\mu A_\nu$ is not a tensor under general coordinate transformations, where ∂_μ is the ordinary partial derivative with respect to x^μ and A_ν are the components of a covector.

[7 marks]

1.6) Show that a stone falling radially into a Schwarzschild black hole of mass M from spatial infinity, with zero initial velocity, moves with the speed of light as it crosses the event horizon ($r = 2M$, in units $G_N = c = 1$), as measured by *nearby observers*.

[7 marks]

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1.7) By using space-time diagrams explain why the result of the Pound-Rebka-Snider experiment cannot be accommodated within the framework of Special Relativity alone. Assume that the Earth is non rotating, and that the observers are stationary relative to each other, as well as to Earth.

[7 marks]

1.8) State clearly the two forms (weak and strong) of the Equivalence Principle of General Relativity.

[7 marks]

SECTION B - Answer TWO questions

2) Consider the three-dimensional space time:

$$ds^2 = dz^2 + e^{\sigma(z)}\eta_{ij}dx^i dx^j \ , \qquad i,j = 1,2$$

where η_{ij} is the two-dimensional Minkowski metric, and $\sigma(z)$ a scalar function of z.

(i) Show that this spacetime can become conformally flat, that is with metric of the form $g_{\alpha\beta} = e^{2\varphi(y)}\eta_{\alpha\beta}, \alpha, \beta = 1,2,3,$ where $\eta_{\alpha\beta}$ denotes the three-dimensional Minkowski metric, after an appropriate coordinate transformation to some coordinates $y^\alpha(z, x^1, x^2)$, $\alpha = 1,2,3$, with $\varphi(y)$ a function of these coordinates.

[8 marks]

(ii) Show that the appropriate *null* geodesic equation in the coordinate frame of (i) - i.e., where the metric has a conformally flat form - reads:

$$\frac{dp^\alpha}{d\lambda} + 2p^\alpha \left(\nabla\varphi \cdot p\right) = 0 \ ,$$

where $p^\mu = dy^\mu/d\lambda$, and λ is the affine parameter parametrizing the geodesic.

[12 marks]

(iii) By multiplying the null geodesic equation of (ii) by $e^{2\varphi(y)}$, and rescaling λ such that $d\lambda' = e^{-2\varphi}d\lambda$, show that

$$dp^\alpha/d\lambda' = 0$$

and thus determine the shape of the null geodesics of (ii).

[10 marks]

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3) Consider the two-dimensional spacetime described by the infinitesimal line element:

$$ds^2 = -dt^2 + e^{2t} dr^2,$$

where t is the time coordinate.

(i) By using an appropriate variational method, or otherwise, compute the Christoffel symbols for the above spacetime, and write down the appropriate geodesics.

[8 marks]

(ii) Compute the independent components of the Riemann tensor in this two dimensional geometry.

[6 marks]

(iii) Show that the components of the Ricci tensor, for this spacetime are:

$$R_{tt} = -1, \qquad R_{rr} = e^{2t}, \qquad R_{tr} = R_{rt} = 0.$$

[6 marks]

(iv) Compute the curvature scalar, $R = g^{\mu\nu} R_{\mu\nu}$, for this spacetime.

[4 marks]

(v) Discuss the evolution of this two-dimensional universe, in particular the asymptotic behaviour of the cosmic acceleration as $t \to \infty$. What do you conclude about the cosmic horizon in this case?

[6 marks]

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4) Einstein's equations for a four-dimensional Friedmann–Robertson–Walker (FRW) universe, with scale factor $a(t)$, assume the form:

$$
\begin{aligned}
-3\frac{\dot{a}^2(t)}{a^2(t)} - 3\frac{k}{a^2(t)} + \Lambda &= -8\pi G_N \rho \ , \\
-2\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)} - \frac{k}{a^2(t)} + \Lambda &= 8\pi G_N p \ , \qquad (II)
\end{aligned}
$$

where k is the usual characteristic parameter of the FRW cosmology, G_N is Newton's constant, Λ is the cosmological constant, ρ is the energy density, and p is the pressure density.

(i) Consider first the case of a Λ dominated FRW Universe, i.e. a Universe in which the Λ term in Einstein's equations is dominant over any other contributions from the matter stress tensor. In that case one may absorb the Λ term in a 'modified' stress tensor which thus has the form:

$$T^{\Lambda}_{\mu\nu} = -g_{\mu\nu}\frac{\Lambda}{8\pi G_N}.$$

where $g_{\mu\nu}$ is the metric of a FRW universe with parameter k:

$$ds^2 = -dt^2 + a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right\}.$$

Does this Λ-dominated Universe constitute an ideal fluid situation? Justify your answer.

[6 marks]

(ii) Assume the general expression for stress tensors in ideal fluids in a frame which moves with four velocity u^μ relative to an inertial observer

$$T^{\text{idealfluid}}_{\mu\nu} = p g_{\mu\nu} + (p+\rho)u_\mu u_\nu$$

and compare component by component with the above definition of $T^{\Lambda}_{\mu\nu}$ in a comoving cosmological frame ($u^0 = 1, u^i = 0$). In this way determine the equation of state of the Λ-dominated FRW Universe, that is determine the function f in the equation $p_\Lambda = f(\rho_\Lambda)$, where p_Λ is the pressure and ρ_Λ is the energy density of this fluid.

[12 marks]

(iii) For a matter dominated Universe you may assume that $\rho \propto a^{-3}$. Use equations (II) to determine the dependence of $a(t)$ asymptotically in the FRW time t, $t \to \infty$, in the cases of open or flat FRW Universes with positive cosmological constant $\Lambda > 0$, where the Universe scale factor is assumed to diverge as $t \to \infty$. Show that such Universes suffer from the problem of eternal acceleration and a cosmic horizon.

[12 marks]

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5) Light in General Relativity follows by definition null geodesics. Consider a three-dimensional space time with Schwarzschild geometry, that is assume $d\phi = 0$, and $\theta \in [0, 2\pi]$, in the respective formulae in the rubric.

(i) Consider *radial* motion of light in this three-dimensional Schwarzschild space time. Work in units for which $G_N = c = 1$. Show that the radial velocity of light in Bookkeeper Schwarzschild coordinates is given by

$$\frac{dr}{dt} = \pm \left(1 - \frac{2M}{r}\right)$$

Explain physically the meaning of the $\pm$ sign in front of this formula.

[8 marks]

(ii) Carry out a similar analysis as in (i) but for *tangential* motion of light in the three-dimensional Schwarzschild geometry. Define and determine the tangential velocity of light in Bookkeeper Schwarzschild coordinates (work in units for which $G_N = c = 1$).

[8 marks]

(iii) Discuss the values of the tangential and radial velocities of light at the horizon $r = 2M$. Identify an apparent paradox that, at first sight, seems to contradict Special Relativity.

[6 marks]

(iv) Compute the velocity of light as measured by nearby *shell* observers at the horizon, $ds_{\text{shell}}/dt_{\text{shell}}$, where $ds^2_{\text{shell}} = dr^2_{\text{shell}} + r^2 d\theta^2|_{\text{shell}}$, and thus resolve the apparent paradox of (iii).

[8 marks]

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