King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/3270 Chaos in Physical Systems

Summer 1999

Time allowed: THREE Hours

**Candidates should answer SIX parts of SECTION A,
and TWO questions from SECTION B.**

Separate answer books must be used for each Section of the paper.

The approximate mark for each part of a question is indicated in square brackets.

**You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.**

TURN OVER WHEN INSTRUCTED

SECTION A – Answer SIX parts of this section

1.1) Find the fixed points of the flow

$$\frac{dx}{dt} = \sin x.$$

[7 marks]

1.2) Convert the following equation into the standard form of a set of coupled first order differential equations:

$$\frac{d^3x}{dt^3} = -bx^2$$

where b is constant.

[7 marks]

1.3) Define stable and unstable manifolds for a fixed point. Hence define a homoclinic point.

[7 marks]

1.4) Use linear stability analysis to study the dynamical behaviour of the one-dimensional system

$$\frac{dx}{dt} = ax - bx^3$$

(a, b being constant) for $a < 0$ and $b > 0$.

[7 marks]

1.5) By considering

$$\frac{dx}{dt} = rx - x^2$$

describe a transcritical bifurcation.

[7 marks]

1.6) State the essential characteristics of chaos.

[7 marks]

1.7) Describe the Ruelle-Takens-Newhouse scenario for the onset of chaos.

[7 marks]

1.8) For the logistic map define the Feigenbaum number. Discuss why the logistic map is relevant for understanding the bifurcations in Rayleigh-Benard convection.

[7 marks]

SECTION B – Answer TWO questions

2) Define box-counting dimension.

[10 marks]

Use this to express the fractal dimension d_L of an attractor for a dissipative system in 3-dimensions which has a positive, negative and zero Lyapunov exponent. Show that d_L is an example of the general form

$$d_L = k + \frac{\sum_{i=1}^{k} \lambda_i}{|\lambda_{k+1}|}$$

where $\lambda_1 > \lambda_2 > \lambda_3 > \cdots > \lambda_n$ and k is the largest integer such that $\sum_{i=1}^{k} \lambda_i > 0$.

[10 marks]

For a particular mapping it is found that an attractor has the following values of Lyapunov exponents:

$$\lambda_1 = 0.67,\ \lambda_2 = -0.70,\ \lambda_3 = -1.36.$$

Calculate the corresponding d_L.

[10 marks]

3) For the following systems of linear mappings $(x, y) \to (x', y')$ classify the stability characteristic of the steady state at $(x, y) = (0, 0)$:

(a)

$$x' = -2y$$
$$y' = x$$

[10 marks]

(b)

$$x' = 3x + 2y$$
$$y' = 4x + y$$

[10 marks]

(c)

$$x' = -4x - 2y$$
$$y' = 3x - y$$

[10 marks]

4) Show that the Henon transform $f(x,y) : R^2 \to R^2$ where

$$f(x,y) = \left(1 + y - ax^2, bx\right)$$

and a, b are constants is invertible if b is non-zero.Write down the transformation that describes the inverse mapping.

[10 marks]

By constructing the determinant of the Jacobian matrix show that the Henon map contracts area if $|b| < 1$.

[10 marks]

Sketch the effect of the Henon transform on the rectangle bounded by the lines $x = 0$, $x = u$, $y = 0$ and $y = v$ in the xy plane.

[10 marks]

5) Consider the iterative map

$$x_{n+1} = \begin{cases} 2x_n, & 0 \leq x_n \leq \frac{1}{2} \\ 2 - 2x_n, & \frac{1}{2} \leq x_n \leq 1 \end{cases}$$

Find the fixed points and classify their stability.

[10 marks]

Show that the map has a period-2 orbit.

[10 marks]

Show that this orbit is unstable.

[10 marks]