King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/3270 Chaos in Physical Systems

Summer 2000

Time allowed: THREE Hours

**Candidates should answer SIX parts of SECTION A,
and TWO questions from SECTION B.**

Separate answer books must be used for each Section of the paper.

The approximate mark for each part of a question is indicated in square brackets.

**You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.**

TURN OVER WHEN INSTRUCTED

SECTION A – Answer SIX parts of this section

1.1) Graph the potential for the system $\dot{x} = -x$ and identify all of the equilibrium points.

[7 marks]

1.2) Give a linear stability analysis of the fixed points of

$$\dot{x} = r - x^2,$$

where r is a real constant.

[7 marks]

1.3) Find the conditions under which it is valid to approximate the equation

$$mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin\theta = \Gamma$$

by its overdamped limit

$$b\dot{\theta} + mgL\sin\theta = \Gamma.$$

[7 marks]

1.4) Solve the linear system

$$\dot{\vec{x}} = A\,\vec{x}$$

where $A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}$.

Graph the phase portrait as a varies from $-\infty$ to ∞, showing the qualitatively different cases.

[7 marks]

1.5) Show that a conservative system cannot have any attracting fixed points.

[7 marks]

1.6) State the Poincaré-Bendixson theorem.

[7 marks]

1.7) Show that the measure of the Cantor set is zero, in the sense that it can be covered by intervals whose total length is arbitrarily small.

[7 marks]

1.8) Show that the map

$$x_{n+1} = x_n + y_{n+1}$$
$$y_{n+1} = y_n + kx_n$$

is area-preserving for all k.

[7 marks]

SECTION B – Answer TWO questions

2) The Maxwell-Bloch equations provide a model for the laser. These equations describe the dynamics of the electric field E, the mean polarization P of the atoms, and the population inversion D:

$$\dot{E} = \kappa\,(P - E)$$
$$\dot{P} = \gamma_1\,(ED - P)$$
$$\dot{D} = \gamma_2\,(\lambda + 1 - D - \lambda EP)$$

where κ is the decay rate in the laser cavity due to beam transmission, γ_1 and γ_2 are decay rates of the atomic polarization and population inversion, respectively, and λ is a pumping energy parameter. The parameter λ is greater than -1; all the other parameters are positive.

Assume $\gamma_1, \gamma_2 >> \kappa$ when $\dot{P} \approx 0$ and $\dot{D} \approx 0$.

Using this condition express P and D in terms of E and thereby derive a first-order equation for the evolution of E.

[10 marks]

Find all the fixed points E^* of E.

[10 marks]

Draw the bifurcation diagram of E^* versus λ.

[10 marks]

3) State the equivalent circuit description of a Josephson junction.

[8 marks]

By considering the behaviour of the circuit, derive the analogy with a damped pendulum.

[8 marks]

Show that a dimensionless formulation of the system has the form

$$\beta \frac{d^2}{d\tau^2}\phi + \frac{d}{d\tau}\phi + \sin\phi = \frac{I}{I_c}$$

where β is the McCumber parameter and ϕ is the phase difference across the Josephson junction. In the overdamped limit $\beta << 1$ for $I < I_c$ show that ϕ goes to a constant for large positive τ.

[14 marks]

4) Consider the decimal shift map on the unit interval given by

$$x_{n+1} = 10x_n \pmod 1$$

As usual mod 1 denotes that only the non-integer part of x is considered.

Draw the graph of the map.

[6 marks]

By writing x_n in decimal form, find all the fixed points.

[6 marks]

Show that the map has periodic points of all periods, and that all them are unstable.

[5 marks]

Show that the map has an infinite number of aperiodic orbits.

[7 marks]

By considering the rate of separation of two nearby orbits, show that the map has sensitive dependence on initial conditions.

[6 marks]

5) In the fundamental biochemical process called glycolysis, living cells obtain energy by breaking down sugar, and this can occur in an oscillatory way. A simple model to describe these oscillations is given by

$$\dot{x} = -x + ay + x^2y \quad \text{and}$$
$$\dot{y} = b - ay - x^2y$$

where x and y are concentrations of adenosine diphosphate and fructose-6-phosphate repectively, and $a, b > 0$ are kinetic parameters.

Define and construct a trapping region for this system.

[25 marks]

Determine whether or not there is a closed orbit inside the trapping region by applying the Poincaré-Bendixson theorem.

[5 marks]