King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/3201 MATHEMATICAL METHODS IN PHYSICS III

Summer 1999

Time allowed: THREE Hours

Candidates should answer SIX parts of SECTION A, and TWO questions from SECTION B.

Separate answer books must be used for each Section of the paper.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

SECTION A – Answer any SIX parts of this section

1.1) A certain analytic function $f(z) = u(x, y) + iv(x, y)$ has a real part

$$u(x, y) = 3x^3 - 9xy^2 - 2xy\,.$$

Use the Cauchy-Riemann equations to determine the imaginary part $v(x, y)$ of this function.

[7 marks]

1.2) Determine all the values of the number $\mathrm{Re}\,(-1-i)^i$.

[7 marks]

1.3) Locate and classify all the singularities in the finite z plane of the function

$$f(z) = \frac{(2z^2 + 5z + 3)\sin z}{z(z^2 - 1)^2}\,.$$

[7 marks]

1.4) Determine the Laurent series for the function

$$f(z) = \frac{z}{(z+1)(z+4)}$$

which is valid in the region $0 < |z + 1| < 3$.

[7 marks]

1.5) Describe the methods that can be used to calculate the residue of a function $f(z)$ at an isolated singularity $z = a$. Determine the residue of the function

$$f(z) = \frac{z^2}{(z+2)^2\,(z+3)}$$

at the point $z = -2$.

[7 marks]

1.6) Use the Bessel function series

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)} \left(\frac{z}{2}\right)^{2m+\nu} ,$$

where $\Gamma(x)$ denotes the gamma function, to derive the relation

$$\frac{d}{dz}\left[z^\nu J_\nu(z)\right] = z^\nu J_{\nu-1}(z) .$$

[7 marks]

1.7) A bead of mass m is constrained to move in the xz plane along a smooth rigid wire which has the shape of the hyperbola $xz = C$, where $0 < x < \infty$ and C is a positive constant. The force of gravity acts in the negative z direction. Derive an expression for the Lagrangian of the system using x as a generalised coordinate.

[7 marks]

1.8) Use the method of Lagrange multipliers to find the extremum value of the function

$$f(x, y, z) = xyz,$$

where the variables x, y, z are subject to the constraint

$$\frac{1}{x} + \frac{5}{y} + \frac{1}{z} = 1.$$

[7 marks]

SECTION B – Answer TWO questions in this section

2) State the *residue theorem* for evaluating contour integrals in complex analysis.

[3 marks]

Use the residue theorem to evaluate the following definite integrals:

$$\text{(a)} \qquad \int_0^{2\pi} \frac{\exp(-i\theta)}{(5-3\sin\theta)^2}\, d\theta \ ,$$

[12 marks]

$$\text{(b)} \qquad \int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\, dx \qquad .$$

[15 marks]

In part (b), justification should be given for the neglect of any contour integral which is not taken along the real axis.

3) A semi-circular stretched membrane of radius a lies in a region of the xy plane with plane polar coordinates $0 \leq \rho \leq a$ and $0 \leq \phi \leq \pi$. The membrane has all its boundary edges clamped in the xy plane. When the membrane is allowed to vibrate freely with small amplitude the vertical displacement ψ of the membrane satisfies the equation

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}\,,$$

where c is a constant. Show that the normal modes of vibration of the membrane are

$$\psi_{m,s}(\rho,\phi,t) = A_{m,s}\, J_m(k_{m,s}\,\rho)\,\sin(m\phi)\,\cos(ck_{m,s}t + B_{m,s})\,,$$

where $m, s = 1, 2, \ldots$, $A_{m,s}$ and $B_{m,s}$ are constants,

$$k_{m,s} = j_{m,s}/a\,,$$

and $\{j_{m,s}; s = 1, 2, \ldots\}$ are the positive zeros of the Bessel function $J_m(z)$.

[20 marks]

Show that the radial part of the normal mode $\psi_{m,s}(\rho,\phi,t)$ satisfies the orthogonality relation

$$\int_0^a J_m\left(j_{m,r}\frac{\rho}{a}\right) J_m\left(j_{m,s}\frac{\rho}{a}\right)\rho\, d\rho = 0\,,$$

where $r, s = 1, 2, \ldots$ and $r \neq s$.

[10 marks]

[It may be assumed that $J_m(z)$ is a solution of the differential equation

$$z^2\, w'' + z\, w' + (z^2 - m^2)w = 0\,.\ ]$$

4) Derive the Hamilton canonical equations of motion for a classical system which has a Lagrangian $L(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t)$ corresponding to n degrees of freedom.

[8 marks]

A particle of mass m is constrained to move on the surface of a smooth cone which has a parametric representation

$$x = \rho\cos\phi\,, \quad y = \rho\sin\phi\,, \quad z = k\rho\,,$$

where $0 < \rho < \infty$, $0 \le \phi \le 2\pi$ and k is a positive constant. No external forces act on the particle. Derive an expression for the Lagrangian of the system. Hence obtain the Lagrange equations of motion.

[10 marks]

Show that the path $\rho = \rho(\phi)$ of the particle satisfies the differential equation

$$(1+k^2)\,\frac{d^2u}{d\phi^2} + u = 0\,,$$

where $u = 1/\rho$.

[7 marks]

Describe a *geometrical* procedure which could be used to construct the path of the particle.

[5 marks]

5) A functional $J : A^2(x_0, x_1) \to R^1$ is defined by

$$J[y] = \int_{x_0}^{x_1} F(x, y, y')\, dx\,,$$

where the function $F(x, y, y')$ has continuous second-order derivatives with respect to all its arguments, R^1 denotes a real number and $y' = dy/dx$. The class $A^2(x_0, x_1)$ of admissible functions consists of all functions $y(x)$ which have a continuous second-order derivative for $x_0 \le x \le x_1$ and have the same fixed end-point values $y(x_0) = y_0$ and $y(x_1) = y_1$. Prove that if $y(x) \in A^2(x_0, x_1)$ gives an extremum to $J[y]$ then it must satisfy the differential equation

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0\,.$$

[11 marks]

Hence show that if $F(x, y, y')$ does not depend explicitly on the variable x, then the extremal function $y(x)$ also satisfies the equation

$$F - y'\frac{\partial F}{\partial y'} = C\,,$$

where C is a constant.

[7 marks]

Determine the extremal function $y(x) \in A^2(0, a)$ for the functional

$$J[y] = \int_0^a y^2\,(1 - y')^2\, dx$$

which passes through the end-points $(x_0, y_0) = (0, 0)$ and $(x_1, y_1) = (a, b)$, where a and b are contants.

[12 marks]