# King's College London

UNIVERSITY OF LONDON

This paper is part of an examination of the College counting towards the award of a degree. Examinations are governed by the College Regulations under the authority of the Academic Board.

B.Sc. EXAMINATION

CP/3201 MATHEMATICAL METHODS IN PHYSICS III

Summer 1998

Time allowed: THREE Hours

**Candidates should answer SIX parts of SECTION A, and TWO questions from SECTION B.**

**Separate answer books must be used for each Section of the paper.**

**The approximate mark for each part of a question is indicated in square brackets.**

**You must not use your own calculator for this paper. Where necessary, a College calculator will have been supplied.**

TURN OVER WHEN INSTRUCTED

## SECTION A – Answer any SIX parts of this section

1.1) A certain analytic function $f(z) = u(x, y) + iv(x, y)$ has an imaginary part

$$v(x, y) = 4xy - 2x + 3y\,.$$

Use the Cauchy-Riemann equations to determine the real part $u(x, y)$ of $f(z)$.

[7 marks]

1.2) Determine all the values of the number $\mathrm{Re}\,(-1+i)^{1+i}$.

[7 marks]

1.3) Locate and classify all the singularities in the finite $z$ plane of the function

$$f(z) = \frac{(z^2 - 3z + 2)(1 - \cos z)}{z^3 (z+1)^2 (z-2)^4}\,.$$

[7 marks]

1.4) Determine the Laurent series for the function

$$f(z) = \frac{1}{(z-2)(z-4)}$$

which is valid in the region $0 < |z - 2| < 2$.

[7 marks]

1.5) Determine the residue of the function

$$f(z) = \frac{1 + \cos z}{(z - \pi)^3}$$

at the point $z = \pi$.

[7 marks]

1.6) Use the Bessel function series

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m + \nu + 1)} \left(\frac{z}{2}\right)^{2m+\nu}\,,$$

where $\Gamma(x)$ denotes the gamma function, to derive the relation

$$\frac{d}{dz}\left[z^\nu J_\nu(z)\right] = z^\nu J_{\nu-1}(z)\,.$$

[7 marks]

1.7) A bead of mass $m$ slides on a frictionless wire which has a parametric representation

$$x = a(\theta - \sin\theta), \quad y = 0, \quad z = a(1 + \cos\theta)\,,$$

where $0 \leq \theta \leq 2\pi$ and $a$ is a positive constant. The force of gravity acts in the negative $z$ direction. Derive an expression for the Lagrangian of the system by using $\theta$ as a generalized coordinate.

[7 marks]

1.8) Use the method of Lagrange multipliers to find the extremum values of the function

$$f(x, y) = xy,$$

where the variables $x$ and $y$ are subject to the constraint

$$x^2 + 4y^2 = 4.$$

[7 marks]

## SECTION B – Answer TWO questions in this section

2) State the *residue theorem* for evaluating contour integrals in complex analysis. Describe the various methods that can be used to calculate residues.

[8 marks]

Use the residue theorem to evaluate the following definite integrals:

$$\text{(a)} \quad \int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}\, d\theta\,,$$

[9 marks]

$$\text{(b)} \quad \int_{-\infty}^{\infty} \frac{\cos x}{(x^2+16)^2}\, dx\,.$$

[13 marks]

In part (b), justification should be given for the neglect of any contour integral which is not taken along the real axis.

3) Derive the Hamilton canonical equations of motion for a classical system which has a Lagrangian $L(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t)$ corresponding to $n$ degrees of freedom.

[8 marks]

A particle of mass $m$ is constrained to move on the surface of a smooth torus which has a parametric representation

$$x = \rho\cos\psi\,, \quad y = \rho\sin\psi\,, \quad z = b\sin\theta\,,$$

where

$$\rho = a + b\cos\theta\,, \quad (a > b > 0)\,,$$

with $0 \le \psi \le 2\pi$ and $0 \le \theta \le 2\pi$. No external forces act on the particle. Derive an expression for the Lagrangian of the system by using $\theta$ and $\psi$ as generalized coordinates. Hence show that the Hamiltonian of the system can be written in the form

$$\mathcal{H} = \frac{1}{2m}\left[\frac{p_\psi^2}{(a+b\cos\theta)^2} + \frac{p_\theta^2}{b^2}\right]\,.$$

[12 marks]

Derive the Hamilton canonical equations of motion for the particle. Show that if the particle moves round the outer equatorial circle ($\theta = 0$), then $\dot{\psi}$ must be a constant of the motion. Investigate the stability of this equatorial motion when a *small* perturbation is made to the angle $\theta$.

[10 marks]

4) A functional $J : A^3(x_0, x_1) \to R^1$ is defined by

$$J[y] = \int_{x_0}^{x_1} F\left(x, y, y', y''\right)\, dx\,,$$

where the function $F(x, y, y', y'')$ has continuous third-order derivatives with respect to all its arguments, $R^1$ denotes a real number, $y' = dy/dx$ and $y'' = d^2y/dx^2$. The class $A^3(x_0, x_1)$ of admissible functions consists of all functions $y(x)$ which have a continuous third-order derivative for $x_0 \leq x \leq x_1$ and have the same fixed end-point values $y(x_0) = y_0$, $y(x_1) = y_1$, $y'(x_0) = y'_0$ and $y'(x_1) = y'_1$. Prove that if $y(x) \in A^3(x_0, x_1)$ gives an extremum to $J[y]$ then it must necessarily satisfy the differential equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0\,.$$

[16 marks]

A train moving in a straight line has to travel a distance $L$ in a time $T$ and must be stationary ($\dot{y} = 0$) at the beginning ($y = 0$, $t = 0$) and the end ($y = L$, $t = T$) of the journey. Determine the motion $y(t)$ for $0 \leq t \leq T$ which gives an extremum value to the *passenger discomfort functional*

$$D[y] = \int_0^T \left(\ddot{y}\right)^2\, \mathrm{d}t\,.$$

Hence obtain the extremum value of the functional $D[y]$.

[14 marks]

5) A particle of mass $m$ and energy $E$ has a wave function $\psi(\rho, \phi, z)$ which satisfies the Schrödinger equation in cylindrical polar coordinates $(\rho, \phi, z)$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0\,,$$

where $k^2 = 2mE/\hbar^2$. The particle is confined inside a closed cylindrical box with $0 \le \rho \le a$, $0 \le \phi \le 2\pi$ and $0 \le z \le a$. On the surface of the box the wave function satisfies the boundary condition $\psi(\rho, \phi, z) = 0$. Use the method of separation of variables to show that the energy eigenfunctions for the particle are

$$\psi_{\nu,s,n}(\rho, \phi, z) = J_\nu\left(\frac{\rho}{a} j_{\nu,s}\right) \sin\left(\frac{n\pi z}{a}\right) e^{\pm i\nu\phi}\,,$$

where $\nu = 0, 1, 2, \ldots$, both $s, n = 1, 2, \ldots$, and $j_{\nu,1}, j_{\nu,2}, \ldots$ are the positive zeros of the Bessel function $J_\nu(z)$.

[20 marks]

Derive a formula for the corresponding energy eigenvalues $E_{\nu,s,n}$ for the particle. Hence calculate the ground-state energy of the particle in terms of the quantity $\hbar^2/(ma^2)$.

[10 marks]

[It may be assumed that $J_\nu(z)$ is a solution of the differential equation

$$x^2\, y'' + x\, y' + (x^2 - \nu^2) y = 0\,,$$

and the smallest positive zero of $J_0(x)$ is $j_{0,1} = 2.40483\ldots$.]