King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/3201 Mathematical Methods in Physics III

Summer 2000

Time allowed: THREE Hours

Candidates must answer SIX parts of SECTION A, and TWO questions from SECTION B.

Separate answer books must be used for each Section of the paper.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College Calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

SECTION A – Answer SIX parts of this section

1.1) Find the real and imaginary parts of the function e^{iz}, where z is a complex number.

[7 marks]

1.2) Use the Cauchy-Riemann equations to determine whether

$$\frac{y - ix}{x^2 + y^2}$$

is an analytic function.

[7 marks]

1.3) Show that the power series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges at all points on the circle of convergence.

[7 marks]

1.4) The function $f(z)$ has a zero of order α at the point $z = z_0$. Calculate the residue of $z\frac{f'(z)}{f(z)}$. Determine the residue of $\varphi(z)\,\frac{f'(z)}{f(z)}$ at z_0, where $\varphi(z)$ denotes an arbitrary function which is regular at z_0.

[7 marks]

1.5) Using the Laurent series

$$e^{\left(\frac{x}{2}\right)\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

show that

$$J_{n-1}(x) - J_{n+1}(x) = 2\frac{\mathrm{d}}{\mathrm{d}x} J_n(x)\,.$$

.

[7 marks]

1.6) Write down the Euler equation for F which makes the integral

$$\int_{x_1}^{x_2} F(y, \dot{y}, x) \, dx$$

an extremum where $\dot{y} = \frac{\mathrm{d}y}{\mathrm{d}x}$. When

$$F(y, \dot{y}, x) = y^2 + \dot{y}^2$$

solve the Euler equation to find the function $y(x)$ which makes the integral stationary.

[7 marks]

1.7) Define the lagrangian of a mechanical system. Construct the lagrangian for a simple pendulum of length l in terms of an angle θ and from the Euler-Lagrange equation obtain the equation of motion for θ, i.e.

$$\ddot{\theta} + \frac{g}{l} \sin\theta = 0.$$

[7 marks]

1.8) Evaluate

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{5 + 4\cos\theta}$$

by showing that it can be written as a contour integral

$$\frac{1}{i} \oint_C \frac{\mathrm{d}z}{(2z+1)(z+2)} \quad ,$$

where C is the unit circle around the origin in the complex plane.

[7 marks]

SECTION B – Answer TWO questions

2) A mass M_2 hangs at one end of a string which passes over a fixed frictionless non-rotating pulley. The other end of this string is fixed to a frictionless non-rotating pulley of mass M_1 over which there is a string carrying masses m_1 and m_2.Let X_1 and X_2 be the distances of masses M_1 and M_2 respectively below the centre of the fixed pulley. Let x_1 and x_2 be the distances of the masses m_1 and m_2 respectively below the centre of the movable pulley.

Show that the kinetic and potential energies, T and V, of the system are

$$T = \frac{1}{2}M_1\dot{X}_1^2 + \frac{1}{2}M_2\dot{X}_2^2 + \frac{1}{2}m_1\left(\dot{X}_1 + \dot{x}_1\right)^2 + \frac{1}{2}m_2\left(\dot{X}_1 + \dot{x}_2\right)^2$$

and

$$V = -M_1gX_1 - M_2gX_2 - m_1g\left(X_1 + x_1\right) - m_2g\left(X_1 + x_2\right).$$

[11 marks]

Construct the lagrangian L of the system including the constraints that X_1+X_2 and $x_1 + x_2$ are both constant, which reflects the fact that the strings are of fixed and unequal length.

[10 marks]

By using the Euler-Lagrange equations show that the acceleration of mass M_2 is

$$-\frac{\left(M_1 - M_2\right)\left(m_1 + m_2\right) + 4m_1m_2}{\left(M_1 + M_2\right)\left(m_1 + m_2\right) + 4m_1m_2}g.$$

[9 marks]

Hint: In terms of generalised co-ordinates q_α the Euler-Lagrange equations have the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial q_\alpha} = 0.$$

3) A slab of material of constant thermal conductivity is located in the region

$$-\frac{a}{2} \leq x \leq \frac{a}{2}$$
$$0 \leq y < \infty.$$

The temperature Φ at the boundaries of the slab are given by

$$\Phi = \begin{cases} T & x = -\frac{a}{2} \\ 2T & x = \frac{a}{2} \\ 0 & y = 0 \end{cases}$$

Show that the mapping

$$w = \sin\left(\frac{\pi z}{a}\right)$$

transforms the region of the slab (when the $x - y$ plane is considered as the complex plane) into the upper half of the w-plane.

[15 marks]

By solving the Laplace equation in the $w-$plane and on using the mapping, prove that in the steady state

$$\begin{aligned}\Phi =& \frac{T}{\pi}\tan^{-1}\left\{\frac{\cos\left(\frac{\pi x}{a}\right)\sinh\left(\frac{\pi y}{a}\right)}{\sin\left(\frac{\pi x}{a}\right)\cosh\left(\frac{\pi y}{a}\right)+1}\right\} \\ &- \frac{2T}{\pi}\tan^{-1}\left\{\frac{\cos\left(\frac{\pi x}{a}\right)\sinh\left(\frac{\pi y}{a}\right)}{\sin\left(\frac{\pi x}{a}\right)\cosh\left(\frac{\pi y}{a}\right)-1}\right\} + 2T\end{aligned}$$

[15 marks]

4) A simple pendulum hangs on a string of length l. Suppose that the length of the pendulum increases at a steady rate, that is $l = l_0 + vt$.On using the equation of motion show that θ, the angular deviation of the pendulum from the vertical, for the case of small oscillations, satisfies the differential equation

$$l\frac{d^2\theta}{dl^2} + 2\frac{d\theta}{dl} + \frac{g}{v^2}\theta = 0.$$

where g is the acceleration of gravity.

[8 marks]

Show that the solution of this equation which satisfies the boundary conditions that $\theta = \theta_0$ and $\frac{d\theta}{dt} = 0$ when $t = 0$ is

$$\theta = \frac{\pi u_0^2\theta_0}{2u}\left(-N_2\left(u_0\right)J_1\left(u\right) + J_2\left(u_0\right)N_1\left(u\right)\right)$$

where $u = \frac{2(gl)^{\frac{1}{2}}}{v}$ and $u_0 = \frac{2(gl_0)^{\frac{1}{2}}}{v}$.

[22 marks]

Hint:

$$\frac{d^2y}{dx^2} + \frac{1-2a}{x}\frac{dy}{dx} + \left[\left(bcx^{c-1}\right)^2 + \frac{a^2 - p^2c^2}{x^2}\right]y = 0$$

has the solution

$$y = x^a\left(AJ_p\left(bx^c\right) + BN_p\left(bx^c\right)\right)$$

where J_p and N_p are Bessel functions of the first and second kind and A and B are constants.

When finding the unknown constants in the solution, the following formulae may be useful:

$$\frac{d}{dx}\left(x^{-p}J_p\left(x\right)\right) = -x^{-p}J_{p+1}\left(x\right)$$
$$\frac{d}{dx}\left(x^{-p}N_p\left(x\right)\right) = -x^{-p}N_{p+1}\left(x\right)$$

and

$$J_p\left(x\right)N_{p+1}\left(x\right) - J_{p+1}\left(x\right)N_p\left(x\right) = -\frac{2}{\pi x}.$$

5) The Fresnel integrals

$$\int_0^u \cos\left(\xi^2\right) \mathrm{d}\xi$$

and

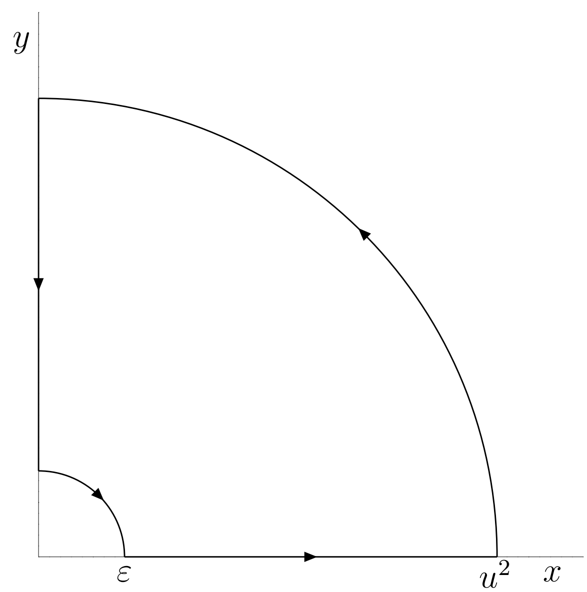
$$\int_0^u \sin\left(\xi^2\right) \mathrm{d}\xi$$

are important in optics. By substituting $x = \xi^2$ show that these integrals may be evaluated as the real and imaginary parts respectively of the integral

$$I = \frac{1}{2} \int_0^{u^2} x^{-\frac{1}{2}} \exp\left(ix\right) \mathrm{d}x.$$

[9 marks]

This integral may be evaluated by considering the corresponding contour integral around the contour in the complex plane as shown below :



The curved portions of the contour are arcs of circles. Show that in the limit $u \to \infty$ and $\varepsilon \to 0$ the only non-zero contributions come from the portions of the contour along the x-axis and y-axis .

[10 marks]

Hence show that

$$\int_0^\infty \cos\left(\xi^2\right) d\xi = \int_0^\infty \sin\left(\xi^2\right) d\xi = \sqrt{\frac{\pi}{8}}.$$

Note: $\Gamma\left(\frac{1}{2}\right) = \pi^{\frac{1}{2}}$.

[11 marks]

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