King's College London

UNIVERSITY OF LONDON

This paper is part of an examination of the College counting towards the award of a degree. Examinations are governed by the College Regulations under the authority of the Academic Board.

B.Sc. EXAMINATION

CP/2380 Electromagnetism

Summer 2005

Time allowed: THREE Hours

Candidates should answer all SIX parts of SECTION A,
and no more than TWO questions from SECTION B.
No credit will be given for answering further questions.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

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Physical Constants

Permittivity of free space	ε_0	=	8.854×10^{-12}	$\mathrm{F\ m^{-1}}$
Permeability of free space	μ_0	=	$4\,\pi \times 10^{-7}$	$\mathrm{H\ m^{-1}}$
Speed of light in free space	c	=	2.998×10^{8}	$\mathrm{m\ s^{-1}}$
Gravitational constant	G	=	6.673×10^{-11}	$\mathrm{N\ m^2\ kg^{-2}}$
Elementary charge	e	=	1.602×10^{-19}	C
Electron rest mass	m_e	=	9.109×10^{-31}	kg
Unified atomic mass unit	m_u	=	1.661×10^{-27}	kg
Proton rest mass	m_p	=	1.673×10^{-27}	kg
Neutron rest mass	m_n	=	1.675×10^{-27}	kg
Planck constant	h	=	6.626×10^{-34}	J s
Boltzmann constant	k_B	=	1.381×10^{-23}	$\mathrm{J\ K^{-1}}$
Stefan-Boltzmann constant	σ	=	5.670×10^{-8}	$\mathrm{W\ m^{-2}\ K^{-4}}$
Gas constant	R	=	8.314	$\mathrm{J\ mol^{-1}\ K^{-1}}$
Avogadro constant	N_A	=	6.022×10^{23}	$\mathrm{mol^{-1}}$
Molar volume of ideal gas at STP		=	2.241×10^{-2}	$\mathrm{m^3}$
One standard atmosphere	P_0	=	1.013×10^{5}	$\mathrm{N\ m^{-2}}$

Maxwell's Equations, and expression for the fields in terms of potentials

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \cdot \mathbf{B} = 0 \qquad \mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Divergence theorem for a vector quantity F and a volume V enclosed by a surface S

$$\int_V \nabla \cdot \mathbf{F}\, dr^3 = \int_S \mathbf{F} \cdot \mathbf{dS}$$

For any vector quantity F and scalar quantity ψ, we have

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$$

$$\nabla \times \nabla \psi = 0$$

If F has the form $\mathbf{F} = \mathbf{F_0} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$ **with $\mathbf{F_0}$ a constant vector, we have**

$$\nabla \times \mathbf{F} = i\mathbf{k} \times \mathbf{F}$$

For a number z << 1 , the following first-order expansion holds: $(1+\mathrm{z})^{\alpha} \cong 1 + \alpha\,\mathrm{z}$

SECTION A – Answer all SIX parts of this section

1.1) Using the divergence theorem and the appropriate Maxwell equation (given in the rubric), derive Coulomb's law expressing the electric field **E** associated to a point charge q**.** [**B** 7 marks]

1.2) A capacitor is made of two identical, parallel coaxial metal disks of surface $S = \pi r^2$, placed in vacuum at distance $h << r$ from each other. The capacitor is charged until electric charges Q and $-Q$ are present on the two disks, corresponding to a potential difference V. Define the capacitance C in terms of Q and V, write an expression for the surface charge density σ on one disk in terms of Q and use the Gauss theorem to express the electric field between the disks as a function of σ and the free space permittivity ε_0. Use your results to show that C is determined by ε_0 and the geometry of the capacitor, independently of the magnitude of the charge. [**B** 7 marks]

1.3) Assuming that the two protons contained in a Helium nucleus are point charges separated by a distance $d_1 = 5 \cdot 10^{-15}$ m, calculate the electrostatic force they exert on each other, stating if it is attractive or repulsive. Limiting yourself to three significant digits accuracy, what is the energy required to take the two protons from d_1 to a distance $d_2 = 5$ m from each other? [**P** 7 marks]

1.4) A permanent magnetic dipole is located at the origin of a Cartesian reference frame, and is oriented along the positive z direction. A second magnetic dipole moment is located on the $z = 0$ plane at a distance d from the first one, and can be oriented along the positive z direction (parallel configuration) or along the negative z direction (antiparallel configuration). State which of these two configurations is energetically more stable, and explain (qualitatively) why. [**P** 7 marks]

1.5) An electro-magnetic signal travelling along the x axis of a Cartesian reference frame can be viewed as a superimposition, or "wave-packet", of monochromatic waves. Discuss how the shape of the wave-packet changes with time while the signal travels (i) in vacuum and (ii) in a uniform medium with a frequency-independent refractive index n. State the speed of the signal in both cases. [**B** 7 marks]

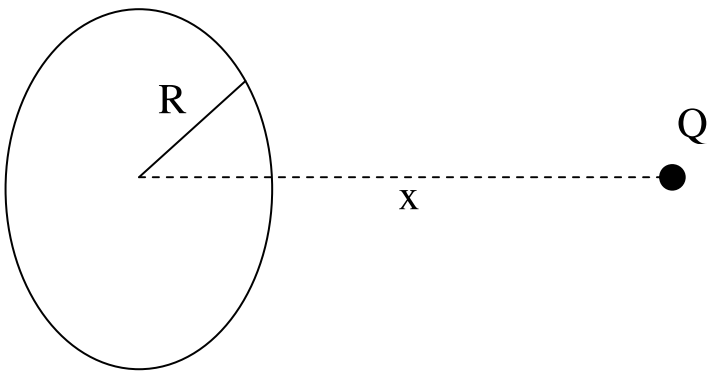
1.6) The fields **E** and **B** can be obtained from the scalar potential Φ and the vector potential **A** (in vacuum, expressions given in the rubric). Using the vector relations in the rubric, show that **B** does not change when the potentials are modified by the (gauge) transformations:

$$\mathbf{A}' = \mathbf{A} + \nabla \psi \quad ; \quad \Phi' = \Phi - \frac{\partial \chi}{\partial t}$$

where ψ and χ are scalar fields. Find a simple relation between ψ and χ so that the electric field **E** is also unchanged by the transformations. [**P** 7 marks]

SECTION B – Answer TWO questions

2) A disk of radius R is charged to a uniform surface charge distribution $\sigma > 0$. A positive point charge Q is located on the axis of the disk at a distance x from it.



(a) Derive an expression for the electrostatic potential generated by the charge on the disk at the location of Q. [**P** 6 marks]

(b) Using the result from (a), show that the total electrostatic force felt by the charge Q is

$$\mathrm{F} = \frac{Q\sigma}{2\varepsilon_0}\left(1 - \frac{x}{\sqrt{R^2 + x^2}}\right) .$$

State the direction of the force on Q. What is the magnitude and direction of the force on the disk? [**P** 6 marks]

(c) Derive approximate expressions to leading order for the force F, assuming that $x >> R$, and expressing the result in terms of the total charge q on the disk. Discuss the physical meaning of the expression found. [**P** 6 marks]

(d) Use the above expression for F to derive an expression for the force felt by the charge Q in the limit $R \rightarrow \infty$, keeping σ fixed. Explain qualitatively why the force is independent of x. [**P** 6 marks]

(e) Use the above expression for F to evaluate the force felt by the charge Q in the limit $x \rightarrow 0$. Compare this result with the result of the previous point (d) and derive an approximate expression to leading order in x for the force in the general case $x << R$. [**P** 6 marks]

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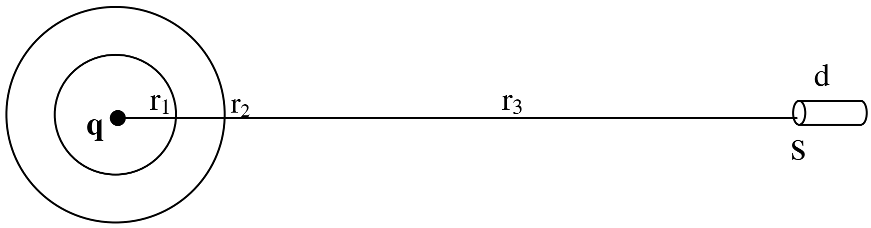
3) A positive point charge q is placed in the middle of a hollow sphere with internal radius r_1 and external radius r_2. The sphere is made of a dielectric material with electric susceptibility $\chi = \varepsilon_r - 1$.

(a) Derive expressions for the electric displacement **D**, the electric field **E,** and the polarization **P** in the three regions defined by $0 < r < r_1$, $r_1 < r < r_2$ and $r > r_2$, specifying both the orientation and magnitude for each of the three vectors. [**P** 8 marks]

(b) Determine the polarization charges which are present on the surfaces of the sphere, stating their sign and the total value for each surface. [**B** 7 marks]

(c) Suppose now that far away from the sphere at a distance $r_3 >> r_2$ we place a cylinder made of the same material as the hollow sphere. The major axis of the cylinder passes through the centre of the sphere. The cylinder has a length $d << r_3$, so that its more distant base is located at $r = r_3+d$ from the centre of the sphere. The area of each base of the cylinder is $S << r_3^2$ (see figure).



Determine the total electric dipole $\mathbf{P}_{TOT}$ of the cylinder, highlighting its dependence on r_3.
In doing this, assume that the polarization **P** in the cylinder can be approximated by a constant vector, taken to be the value of **P** at r_3 (i.e., at the closer base). Also, assume that polarization effects in the cylinder do not perturb the charge distribution in the sphere.
[**P** 8 marks]

(d) Determine the total polarization charge of each base of the cylinder highlighting its dependence on r_3. [**P** 7 marks]

4)

(a) Starting from the Maxwell equations in vacuum show that the magnetic field **B** obeys a wave equation of the form

$$\nabla^2 \mathbf{B} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{B} = 0$$

[**B** 6 marks]

(b) Write down the magnetic field vector **B**(**r**,t) as a plane-wave, using the complex notation and expressing the dependence on the position vector **r** and time t in terms of the wave vector **k** and the angular frequency ω. Show that **B**(**r**,t) written in this way is a solution of the wave equation of part (a) if $\omega = k\,c$, where $k = |\mathbf{k}|$ is the modulus of the wave vector. [**P** 5 marks]

(c) Show that in the plane-wave expression for **B**(**r**,t) k and ω are related to the wavelength λ and the frequency ν through the following relations

$$\mathrm{k} = \frac{2\pi}{\lambda} \quad ; \quad \omega = 2\pi\nu$$

Show that this implies that $\lambda\,\nu = c$. [**B** 4 marks]

(d) Using the Maxwell equation $\nabla \cdot \mathbf{B} = 0$, show that the plane wave solution constructed in part (b) is a *transverse* wave, i.e. that $\mathbf{B} \cdot \mathbf{k} = 0$.

[**P** 4 marks]

(e) Using the Maxwell equation for $\nabla \cdot \mathbf{E}$, and the expression for the electric field **E**(**r**,t) as a function of the vector potential **A**(**r**,t) and the scalar potential Φ(**r**,t), both contained in the rubric, prove that

$$\nabla^2 \Phi + \frac{\partial}{\partial t}\nabla \cdot \mathbf{A} = -\frac{\rho}{\varepsilon_0}$$

[**B** 4 marks]

(f) In a vacuum, the vector potential **A**(**r**,t) can be chosen so that $\nabla \cdot \mathbf{A} = 0$. Prove that the equation in part (e) then reduces, in a vacuum, to Laplace's equation $\nabla^2\Phi = 0$. Hence show that for a constant scalar potential

$$\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{A} \ .$$

Show that if **A**(**r**,t) is a transverse plane-wave, then **E**(**r**,t) is also a transverse plane-wave. By computing **B** as a function of **A**, show that the ordered set of vectors **k**, **E**, **B** constituting the electromagnetic wave is a right handed set of mutually orthogonal vectors [**P** 7marks]