King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/2380 Electromagnetism

Summer 2004

Time allowed: THREE Hours

Candidates should answer no more than SIX parts of SECTION A, and no more than TWO questions from SECTION B.
No credit will be given for answering further questions.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

Physical Constants

Permittivity of free space	$\epsilon_0 =$	8.854×10^{-12}	$\mathrm{F\,m^{-1}}$
Permeability of free space	$\mu_0 =$	$4\pi \times 10^{-7}$	$\mathrm{H\,m^{-1}}$
Speed of light in free space	$c =$	2.998×10^{8}	$\mathrm{m\,s^{-1}}$
Gravitational constant	$G =$	6.673×10^{-11}	$\mathrm{N\,m^2\,kg^{-2}}$
Elementary charge	$e =$	1.602×10^{-19}	C
Electron rest mass	$m_\mathrm{e} =$	9.109×10^{-31}	kg
Unified atomic mass unit	$m_\mathrm{u} =$	1.661×10^{-27}	kg
Proton rest mass	$m_\mathrm{p} =$	1.673×10^{-27}	kg
Neutron rest mass	$m_\mathrm{n} =$	1.675×10^{-27}	kg
Planck constant	$h =$	6.626×10^{-34}	$\mathrm{J\,s}$
Boltzmann constant	$k_\mathrm{B} =$	1.381×10^{-23}	$\mathrm{J\,K^{-1}}$
Stefan-Boltzmann constant	$\sigma =$	5.670×10^{-8}	$\mathrm{W\,m^{-2}\,K^{-4}}$
Gas constant	$R =$	8.314	$\mathrm{J\,mol^{-1}\,K^{-1}}$
Avogadro constant	$N_\mathrm{A} =$	6.022×10^{23}	$\mathrm{mol^{-1}}$
Molar volume of ideal gas at STP	$=$	2.241×10^{-2}	$\mathrm{m^3}$
One standard atmosphere	$P_0 =$	1.013×10^{5}	$\mathrm{N\,m^{-2}}$

The gradient of a scalar function f in spherical coordinates (r, θ, ϕ) is

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi} \right)$$

For any vector quantity $\vec{F}$ and scalar quantity f, we have

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{F}) = \vec{\nabla}(\vec{\nabla}.\vec{F}) - \nabla^2 \vec{F}$$

$$\vec{\nabla} \times \vec{\nabla} f = \vec{0}$$

$$\vec{\nabla}.(\vec{\nabla} \times \vec{F}) = 0$$

Maxwell's equations are, in the vacuum:

$$\vec{\nabla}.\vec{B} = 0$$

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

SECTION A – Answer SIX parts of this section

1.1) In a linear isotropic dielectric the relative permittivity is $\varepsilon_r = 2$ and the electric field has magnitude $E = 10^6$ V m^{-1}. Give the values of the susceptibility, the magnitude of the polarization and the displacement field at this point of the dielectric.

[7 marks]

1.2) The magnetic field generated by a circular loop with current I has magnitude $B = \mu_0 I/(2R)$ at its centre, where R is the radius of the loop. Using this result, compute the magnetic field at the centre of a disc rotating around its axis of symmetry with angular velocity ω, when the disc has a uniform surface-charge density σ_S on its upper side.

[7 marks]

1.3) An electric dipole $\vec{p}$ is located at the origin of the coordinates, with its axis parallel to the z direction. The electric potential generated at the position $\vec{r}$ is

$$\Phi = \frac{\vec{p}.\vec{r}}{4\pi\varepsilon_0 r^3}.$$

Compute, in spherical coordinates, the components of the electric field at the position $\vec{r}$.

[7 marks]

1.4) A capacitor is formed out of two identical parallel plates of surface area 0.2 cm^2 and separation 0.1 mm. The capacitor contains a dielectric with relative permittivity $\varepsilon_r = 3$. A voltage $U = 230$ V is applied across its electrodes. Ignoring edge effects, compute its capacitance, as well as the surface charge density on its plates.

[7 marks]

1.5) Starting from the continuity equation $\partial\rho/\partial t + \vec{\nabla}.\vec{j} = 0$, derive the conservation of electric charge in electromagnetic processes.

[7 marks]

1.6) The energy of a static charge distribution of density ρ, generating the potential Φ, is given by

$$W = \frac{1}{2}\int d^3x\rho\Phi,$$

where the integral is taken over the whole three-dimensional space. Using the appropriate Maxwell equation, show that

$$W = \frac{\varepsilon_0}{2}\int d^3x|\vec{E}|^2,$$

where $\vec{E}$ is the electric field deriving from the potential Φ.

[7 marks]

1.7) Sketch a hysteresis loop, B versus H, describing a ferromagnet and comment on the physical meaning of this curve.

[7 marks]

1.8) Give an expression for the fields $(\vec{E}, \vec{B})$ as functions of the potentials $(\Phi, \vec{A})$ and, using Maxwell equations and the vector identities in the rubric, show that the simultaneous transformations

$$\vec{A} \to \vec{A} + \vec{\nabla}\Lambda \qquad \Phi \to \Phi - \frac{\partial\Lambda}{\partial t}$$

leave $\vec{E}$ and $\vec{B}$ invariant.

[7 marks]

SECTION B – Answer TWO questions

2)

a) Explain why an electrically charged conductor at equilibrium has its electric charges on its surface and give the normal component of the electric field on the surface of the conductor in terms of the surface-charge density σ.

[4 marks]

b) Find the electric field outside and inside a spherical conductor of radius R and charge Q.

[8 marks]

c) Determine the electrostatic energy corresponding to the charge distribution in b).

[5 marks]

d) Find the electric field outside and inside a ball of radius R and charge Q distributed uniformly throughout the ball.

[8 marks]

e) Determine the electrostatic energy corresponding to the charge distribution in d) and compare it to the energy found in c).

[5 marks]

3)

a) A loop with current I generates a magnetic field $\vec{B}$ given by the Biot Savart law:

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l} \times \vec{r}}{r^3}$$

With the aid of a diagram, explain the significance of the quantities appearing in this integral.

[5 marks]

b) A circular loop of radius R is in the plane (x, y), centered on the origin, and a current I flows through it. A point M of the z-axis is labelled by the angle θ between $\hat{z}$, the unit vector along the z-axis, and $\vec{MA}$, where A is a point of the loop. Show that the magnetic field generated at such a point M is

$$\vec{B} = \frac{\mu_0 I}{2R} (\sin\theta)^3 \hat{z}.$$

[15 marks]

c) A solenoid of length L has N complete turns of wire per unit length and has the z-axis as its axis of symmetry. The solenoid has its ends at $z = \pm L/2$ and A denotes a point of the solenoid located at one of its ends. When the current I flows through the solenoid, show that the magnetic field at the centre O of the solenoid is given by

$$\vec{B}_0 = \mu_0 N I \cos\theta_0 \hat{z},$$

where θ_0 is the angle between $\hat{z}$ and $\vec{OA}$.

[10 marks]

4) A circular conducting disc of thickness h and radius R is in a time varying magnetic field perpendicular to the disc, as shown in the figure.

a) Describe the physical process which leads to heat flowing out of the disc.

[5 marks]

b) The magnetic field is uniform and has magnitude $B = B_0 \cos(\omega t)$, where t is the time and ω a constant angular frequency. Using Stokes' theorem and the appropriate Maxwell equation, show that the electric field generated at a distance r from the centre of the disc has magnitude

$$E = \omega \frac{B_0}{2} r \sin(\omega t).$$

What is the direction of $\vec{E}$?

[10 marks]

c) The disc has conductivity γ, i.e. the current is $\vec{j} = \gamma \vec{E}$. The power transferred to the charges per unit volume is $P = \vec{j}.\vec{E}$. Show that the time-average of the power radiated by Joule's effect over one period is

$$\gamma \omega^2 B_0^2 \frac{\pi}{16} h R^4.$$

[10 marks]

d) The disc is replaced by N identical smaller discs of the same height, but area a, occupying together the volume of the initial disc. Relate the area a to the area of the original disc and show that the power radiated is N times smaller than the one computed in c).

[5 marks]

5)

a) Starting from Maxwell's equations in the vacuum, show that the electric field satisfies the wave equation

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}.$$

[6 marks]

b) Assuming a solution of the wave equation obtained in a) of the form

$$\vec{E} = \vec{E}_0 \exp(-i\omega t + i\vec{k}.\vec{r}),$$

show that the phase and group velocities are the same and are equal to the speed of light in vacuum.

[7 marks]

c) Compute the magnetic field $\vec{B}$ corresponding to the time varying field $\vec{E}$ and show that $\vec{B} = \vec{k} \times \vec{E}/\omega$.

[7 marks]

d) Define and compute the Poynting vector $\vec{S}$ for the real fields $\vec{E}_r, \vec{B}_r$ (which are the real parts of $\vec{E}, \vec{B}$ respectively), as well as its time-average over one period. Using Gauss' theorem, explain the physical meaning of the equation

$$\frac{\partial u}{\partial t} + \vec{\nabla}.\vec{S} = 0,$$

where

$$u = \frac{\varepsilon_0}{2} E_r^2 + \frac{1}{2\mu_0} B_r^2.$$

[10 marks]

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